A GENERALIZED FANNES’ INEQUALITY

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ABSTRACT. We axiomatically characterize the Tsallis entropy of a finite quantum system. In addition, we derive a continuity property of Tsallis entropy. This gives a generalization of the Fannes’ inequality.

Key words and phrases: Uniqueness theorem, continuity property, Tsallis entropy and Fannes’ inequality.

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1. INTRODUCTION WITH UNIQUENESS THEOREM OF TSALLIS ENTROPY

Three or four decades ago, a number of researchers investigated some extensions of the Shannon entropy [1]. In statistical physics, the Tsallis entropy, defined in [10] by

\[ H_q(X) \equiv \sum_x \frac{(p(x))^q - p(x)}{1 - q} = \sum_x \eta_q (p(x)) \]

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with one parameter \( q \in \mathbb{R}^+ \) as an extension of Shannon entropy \( H_1(X) = -\sum_x p(x) \log p(x) \), for any probability distribution \( p(x) \equiv p(X = x) \) of a given random variable \( X \), where \( q \)-entropy function is defined by \( \eta_q(x) \equiv -x^q \ln_q x = \frac{x^q - x}{1 - q} \) and the \( q \)-logarithmic function \( \ln_q x \equiv \frac{x^{1-q} - 1}{1-q} \) is defined for \( q \geq 0, q \neq 1 \) and \( x \geq 0 \).

The Tsallis entropy \( H_q(X) \) converges to the Shannon entropy \( -\sum_x p(x) \log p(x) \) as \( q \to 1 \). See [5] for fundamental properties of the Tsallis entropy and the Tsallis relative entropy. In the previous paper [6], we gave the uniqueness theorem for the Tsallis entropy for a classical system, introducing the generalized Faddeev’s axiom. We briefly review the uniqueness theorem for the Tsallis entropy below.

The function \( I_q(x_1, \ldots, x_n) \) is assumed to be defined on \( n \)-tuple \((x_1, \ldots, x_n)\) belonging to

\[
\Delta_n \equiv \left\{ (p_1, \ldots, p_n) \mid \sum_{i=1}^n p_i = 1, \ p_i \geq 0 \ (i = 1, 2, \ldots, n) \right\}
\]

and to take values in \( \mathbb{R}^+ \equiv [0, \infty) \). Then we adopted the following generalized Faddeev’s axiom.

**Axiom 1. (Generalized Faddeev’s axiom)**

(F1) **Continuity:** The function \( f_q(x) \equiv I_q(x, 1 - x) \) with parameter \( q \geq 0 \) is continuous on the closed interval \([0, 1]\) and \( f_q(x_0) > 0 \) for some \( x_0 \in [0, 1] \).

(F2) **Symmetry:** For arbitrary permutation \( \{x'_k\} \in \Delta_n \) of \( \{x_k\} \in \Delta_n \),

\[
(1.1) \quad I_q(x_1, \ldots, x_n) = I_q(x'_1, \ldots, x'_n).
\]

(F3) **Generalized additivity:** For \( x_n = y + z, \ y \geq 0 \) and \( z > 0 \),

\[
(1.2) \quad I_q(x_1, \ldots, x_{n-1}, y, z) = I_q(x_1, \ldots, x_n) + x_n^q I_q \left( \frac{y}{x_n}, \frac{z}{x_n} \right).
\]

**Theorem 1.1** ([6]). The conditions (F1), (F2) and (F3) uniquely give the form of the function \( I_q: \Delta_n \to \mathbb{R}^+ \) such that

\[
(1.3) \quad I_q(x_1, \ldots, x_n) = \mu_q H_q(x_1, \ldots, x_n),
\]

where \( \mu_q \) is a positive constant that depends on the parameter \( q > 0 \).

If we further impose the normalized condition on Theorem [1.1] it determines the entropy of type \( \beta \) (the structural \( a \)-entropy), (see [1] p. 189)).

**Definition 1.1.** For a density operator \( \rho \) on a finite dimensional Hilbert space \( \mathcal{H} \), the Tsallis entropy is defined by

\[
S_q(\rho) \equiv \frac{\text{Tr}[\rho^q - \rho]}{1 - q} = \text{Tr}[\eta_q(\rho)],
\]

with a nonnegative real number \( q \).

Note that the Tsallis entropy is a particular case of \( f \)-entropy [11]. See also [9] for a quasi-entropy which is a quantum version of \( f \)-divergence [3].

Let \( T_q \) be a mapping on the set \( S(\mathcal{H}) \) of all density operators to \( \mathbb{R}^+ \).
Axiom 2. We give the postulates which the Tsallis entropy should satisfy.

(T1) Continuity: For \( \rho \in S(H) \), \( T_q(\rho) \) is a continuous function with respect to the 1-norm \( \|\cdot\|_1 \).

(T2) Invariance: For unitary transformation \( U \), \( T_q(U^* \rho U) = T_q(\rho) \).

(T3) Generalized mixing condition: For \( \rho = \bigoplus_{k=1}^{n} \lambda_k \rho_k \) on \( H = \bigoplus_{k=1}^{n} H_k \), where \( \lambda_k \geq 0 \), \( \sum_{k=1}^{n} \lambda_k = 1 \), \( \rho_k \in S(H_k) \), we have the additivity:

\[
T_q(\rho) = \sum_{k=1}^{n} \lambda_k^q T_q(\rho_k) + T_q(\lambda_1, \ldots, \lambda_n),
\]

where \( (\lambda_1, \ldots, \lambda_n) \) represents the diagonal matrix \( (\lambda_k \delta_{kj})_{k,j=1,\ldots,n} \).

**Theorem 1.2.** If \( T_q \) satisfies Axiom 2 then \( T_q \) is uniquely given by the following form

\[
T_q(\rho) = \mu_q S_q(\rho),
\]

with a positive constant number \( \mu_q \) depending on the parameter \( q > 0 \).

**Proof.** Although the proof is quite similar to that of Theorem 2.1 in [3], we present it for readers’ convenience. From (T2) and (T3), we have

\[
T_q(\lambda_1, \lambda_2) = \lambda_1^q T_q(1) + \lambda_2^q T_q(1) + T_q(\lambda_1, \lambda_2),
\]

which implies \( T_q(1) = 0 \). Moreover, by (T2) and (T3), when \( p_n \neq 1 \), we have

\[
T_q(p_1, \ldots, p_{n-1}, \lambda p_n; (1 - \lambda) p_n)
= p_n^q T_q(\lambda, 1 - \lambda) + (1 - p_n)^q T_q\left(\frac{p_1}{1 - p_n}, \ldots, \frac{p_{n-1}}{1 - p_n}\right) + T_q(p_n, 1 - p_n)
\]

and

\[
T_q(p_1, \ldots, p_{n-1}, p_n) = p_n^q T_q(1) + (1 - p_n)^q T_q\left(\frac{p_1}{1 - p_n}, \ldots, \frac{p_{n-1}}{1 - p_n}\right) + T_q(p_n, 1 - p_n).
\]

From these equations, we have

\[
T_q(p_1, \ldots, p_{n-1}, \lambda p_n; (1 - \lambda) p_n) = T_q(p_1, \ldots, p_{n-1}, p_n) + p_n^q T_q(\lambda, 1 - \lambda).
\]

If we set \( \lambda p_n = y \) and \( (1 - \lambda) p_n = z \) in (1.4), then for \( p_n = y + z \neq 0 \) we have

\[
T_q(p_1, \ldots, p_{n-1}, y, z) = T_q(p_1, \ldots, p_{n-1}, p_n) + p_n^q T_q\left(\frac{y}{p_n}, \frac{z}{p_n}\right).
\]

Then for any \( x, y, z \in \mathbb{R} \) such that \( 0 \leq x, y < 1, 0 < z \leq 1 \) and \( x + y + z = 1 \), we have

\[
T_q(x, y, z) = T_q(x, y + z) + (y + z)^q T_q\left(\frac{y}{y + z}, \frac{z}{y + z}\right)
= T_q(y, x + z) + (x + z)^q T_q\left(\frac{x}{x + z}, \frac{z}{x + z}\right).
\]

If we set \( t_q(x) \equiv T_q(x, 1 - x) \), then we have

\[
t_q(x) + (1 - x)^q t_q\left(\frac{y}{1 - x}\right) = t_q(y) + (1 - y)^q t_q\left(\frac{x}{1 - y}\right).
\]
Taking $x = 0$ and some $y > 0$, we have $T_q(0, 1) = t_q(0) = 0$ for $q \neq 0$. Again setting $\lambda = 0$ in (1.4) and using (T2), we have the reducing condition

$$ T_q(p_1, \ldots, p_n, 0) = T_q(p_1, \ldots, p_n). $$

Thus $T_q$ satisfies all conditions of the generalized Faddeev’s axiom (F1), (F2) and (F3). Therefore we can apply Theorem 1.1 so that we obtain $T_q(\lambda_1, \ldots, \lambda_n) = \mu_q H_q(\lambda_1, \ldots, \lambda_n)$. Thus we have $T_q(\rho) = \mu_q S_q(\rho)$, for density operator $\rho$. \hfill \square

**Remark 1.3.** For the special case $q = 0$ in the above theorem, we need the reducing condition as an additional axiom.

### 2. A CONTINUITY OF TSALLIS ENTROPY

We give a continuity property of the Tsallis entropy $S_q(\rho)$. To do so, we state a few lemmas.

**Lemma 2.1.** For a density operator $\rho$ on the finite dimensional Hilbert space $\mathbf{H}$, we have

$$ S_q(\rho) \leq \ln_q d, $$

where $d = \dim \mathbf{H} < \infty$.

**Proof.** Since we have $\ln_q z \leq z - 1$ for $q \geq 0$ and $z \geq 0$, we have $\frac{x - x^q y^{1-q}}{1-q} \geq x - y$ for $x \geq 0$, $y \geq 0$, $q \geq 0$ and $q \neq 1$. Therefore the Tsallis relative entropy $[5]$

$$ D_q(\rho|\sigma) = \frac{\text{Tr}[\rho - \rho^{q} \sigma^{1-q}]}{1-q} $$

for two commuting density operators $\rho$ and $\sigma$, $q \geq 0$ and $q \neq 1$, is nonnegative. Then we have $0 \leq D_q(\rho|\sigma I) = -d^{-1} (S_q(\rho) - \ln_q d)$. Thus we have the present lemma. \hfill \square

**Lemma 2.2.** If $f$ is a concave function and $f(0) = f(1) = 0$, then we have

$$ |f(t + s) - f(t)| \leq \max \{f(s), f(1-s)\} $$

for any $s \in [0, 1/2]$ and $t \in [0, 1]$ satisfying $0 \leq s + t \leq 1$.

**Proof.**

(1) Consider the function $r(t) = f(s) - f(t + s) + f(t)$. Then $r'(t) \geq 0$ since $f'$ is a monotone decreasing function. Thus we have $r(t) \geq 0$ by $r(0) = 0$. Therefore $f(t + s) - f(t) \leq f(s)$.

(2) Consider the function $l(t) = f(t + s) - f(t) + f(1-s)$. Then $l'(t) \leq 0$. Thus we have $l(t) \geq 0$ by $l(1-s) = 0$. Therefore $-f(1-s) \leq f(t + s) - f(t)$.

Thus we have the present lemma. \hfill \square

**Lemma 2.3.** For any real number $u, v \in [0, 1]$ and $q \in [0, 2]$, if $|u - v| \leq \frac{1}{2}$, then $|\eta_q(u) - \eta_q(v)| \leq \eta_q(|u - v|)$.
Proof. Since $\eta_q$ is a concave function with $\eta_q(0) = \eta_q(1) = 0$, we have
\[ |\eta_q(t + s) - \eta_q(t)| \leq \max \{ \eta_q(s), \eta_q(1 - s) \} \]
for $s \in [0, 1/2]$ and $t \in [0, 1]$ satisfying $0 \leq t + s \leq 1$, by Lemma 2.2. Here we set
\[ h_q(s) \equiv \eta_q(s) - \eta_q(1 - s), \quad s \in [0, 1/2], \quad q \in [0, 2]. \]
Then we have $h_q(0) = h_q(1/2) = 0$ and $h_q''(s) \leq 0$ for $s \in [0, 1/2]$. Therefore we have $h_q(s) \geq 0$, which implies
\[ \max \{ \eta_q(s), \eta_q(1 - s) \} = \eta_q(s). \]
Thus we have the present lemma by letting $u = t + s$ and $v = t$. \hfill $\Box$

**Theorem 2.4.** For two density operators $\rho_1$ and $\rho_2$ on the finite dimensional Hilbert space $H$ with $\dim H = d$ and $q \in [0, 2]$, if $\|\rho_1 - \rho_2\|_1 \leq q^{1/(1-q)}$, then
\[ |S_q(\rho_1) - S_q(\rho_2)| \leq \|\rho_1 - \rho_2\|_1^q \ln_q d + \eta_q(\|\rho_1 - \rho_2\|_1), \]
where we denote $\|A\|_1 \equiv \text{Tr} [(A^*A)^{1/2}]$ for a bounded linear operator $A$.

Proof. Let $\lambda_1^{(1)} \geq \lambda_2^{(1)} \geq \cdots \geq \lambda_d^{(1)}$ and $\lambda_1^{(2)} \geq \lambda_2^{(2)} \geq \cdots \geq \lambda_d^{(2)}$ be eigenvalues of two density operators $\rho_1$ and $\rho_2$, respectively. (The degenerate eigenvalues are repeated according to their multiplicity.) We set $\varepsilon \equiv \sum_{j=1}^d \varepsilon_j$ and $\tilde{\varepsilon}_j \equiv \left| \lambda_j^{(1)} - \lambda_j^{(2)} \right|$. Then we have
\[ \varepsilon_j \leq \varepsilon \leq \|\rho_1 - \rho_2\|_1 \leq q^{1/(1-q)} \leq \frac{1}{2} \]
by Lemma 1.7 of [8]. Applying Lemma 2.3, we have
\[ |S_q(\rho_1) - S_q(\rho_2)| \leq \sum_{j=1}^d \left| \eta_q \left( \lambda_j^{(1)} \right) - \eta_q \left( \lambda_j^{(2)} \right) \right| \leq \sum_{j=1}^d \eta_q(\varepsilon_j). \]
By the formula $\ln_q(xy) = \ln_q x + x^{1-q} \ln_q y$, we have
\[ \sum_{j=1}^d \eta_q(\varepsilon_j) = -\sum_{j=1}^d \varepsilon_j^q \ln_q \varepsilon_j \]
\[ = \varepsilon \left\{ -\sum_{j=1}^d \frac{\varepsilon_j}{\varepsilon} \ln_q \left( \frac{\varepsilon_j}{\varepsilon} \right) \right\} \]
\[ = \varepsilon \left\{ -\sum_{j=1}^d \frac{\varepsilon_j}{\varepsilon} \ln_q \frac{\varepsilon_j}{\varepsilon} - \sum_{j=1}^d \frac{\varepsilon_j}{\varepsilon} \left( \frac{\varepsilon_j}{\varepsilon} \right)^{1-q} \ln_q \frac{\varepsilon_j}{\varepsilon} \right\} \]
\[ = \varepsilon^q \sum_{j=1}^d \eta_q \left( \frac{\varepsilon_j}{\varepsilon} \right) + \eta_q(\varepsilon) \]
\[ \leq \varepsilon^q \ln_q d + \eta_q(\varepsilon). \]
In the above inequality, Lemma 2.1 was used for $p = (\varepsilon_1/\varepsilon, \ldots, \varepsilon_d/\varepsilon)$. Therefore we have
\[ |S_q(\rho_1) - S_q(\rho_2)| \leq \varepsilon^q \ln_q d + \eta_q(\varepsilon). \]
Now $\eta_q(x)$ is a monotone increasing function on $x \in [0, q^{1/(1-q)}]$. In addition, $x^q$ is a monotone increasing function for $q \in [0, 2]$. Thus we have the present theorem.

By taking the limit as $q \to 1$, we have the following Fannes’ inequality (see pp.512 of [7], also [4, 2, 8]) as a corollary, since $\lim_{q \to 1} q^{1/(1-q)} = \frac{1}{e}$.

**Corollary 2.5.** For two density operators $\rho_1$ and $\rho_2$ on the finite dimensional Hilbert space $H$ with $\dim H = d < \infty$, if $\|\rho_1 - \rho_2\|_1 \leq \frac{1}{e}$, then

$$|S_1(\rho_1) - S_1(\rho_2)| \leq \|\rho_1 - \rho_2\|_1 \ln d + \eta_1(\|\rho_1 - \rho_2\|_1),$$

where $S_1$ represents the von Neumann entropy $S_1(\rho) = \text{Tr}[\eta_1(\rho)]$ and $\eta_1(x) = -x \ln x$.

**REFERENCES**


