A Note on Reversal Complexities of Real-Time Counter Machines

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SUMMARY This paper gives a hierarchical property on the number of reversals of real-time counter machines. That is, we show that for any \( k \geq 1 \), a real-time counter machine with \( 2k+1 \) reversals is more powerful than one with \( k \) reversals.

1. Introduction

In Ref. (1), Chan investigated some properties of counter machines with non-constant reversal-bounded counters, and showed that \( n^r \)-reversal-bounded one-way deterministic counter machines are more powerful than \( n^r \) reversal bounded ones, where \( r_1 > r_2 > 0 \).

This short paper investigates a hierarchical properties of real-time one counter machines with constant reversal and shows that for each \( k \geq 1 \), a real-time counter machine with \( 2k+1 \) reversals is more powerful than one with \( k \) reversals.

2. Preliminaries

A counter machine is a pushdown machine whose pushdown store operates as counter, i.e. has a single-letter alphabet. In this paper, we consider a real-time counter machine with constant reversal-bounded counter. A machine is real-time if it reads a new input symbol in every step, and the machine stops immediately after reading the endmarker.

We use the following notations. For each \( k \geq 1 \), \( \text{NRTRBCM}(k) \) (\( \text{DRTRBCM}(k) \)) denotes a nondeterministic (deterministic) real-time \( k \) reversal bounded counter machine. For each \( X \subseteq \{D, N\} \), we let \( \langle \text{XRTRBCM}(k) \rangle = \{T \mid T \text{ is accepted by some XRTRBCM}(k)\} \).

3. Result

In Ref. (2), Duriš and Galil introduce crossing sequences on the working tape of a real-time Turing machine. Note that any crossing sequence at a given boundary on the working tape defines a partition of the input string \( x \) into segments \( x_1 \cdots x_n \). Each time this boundary is crossed, a new segment is determined.

The following lemma is easily proved by using a modification of the proof of Lemma 1 in Ref. (2).

(Lemma 1) Assume there are two accepting computations by a real-time counter machine \( M \) on inputs \( x = x_1 \cdots x_k \) and \( y = y_1 \cdots y_k \) with two identical crossing sequences, and that the \( k \) segments of \( x \) and \( y \) are defined by each crossing sequence. (Note that, the boundaries which determine crossing sequences on inputs \( x \) and \( y \) are not always located at the same place on the working tape.) Then \( M \) also accepts \( x_1 y_1 x_2 y_2 \cdots (x_k y_k x_k \cdots) \).

(Theorem 1) For each \( X \subseteq \{D, X\} \) and each \( k \geq 1 \),

\[ \langle \text{XRTRBCM}(k) \rangle \subseteq \langle \text{XRTRBCM}(2k+1) \rangle . \]

(Proof) For each \( r \geq 1 \), let

\[ S(r) = \{0^{i_1}1^{i_2}2^{i_3}3^{i_4} \cdots 20^n10^m | \forall i (1 \leq i \leq r) [n, i] \geq 1 \}. \]

We can easily see that \( S(k+1) \subseteq \langle \text{DRTRBCM}(2k+1) \rangle \). We then show that \( S(k+1) \notin \langle \text{NRTRBCM}(k) \rangle \). We assure, to the contrary, that \( S(k+1) \) is accepted by an \( \text{NRTRBCM}(k) \) machine with a state set \( Q \). Choose a sufficiently large \( n \) so that

\[ \langle 1 \rangle \ 
\frac{n}{k+1} - \frac{Q(Q+1) + (Q+1) \cdot |Q|^{k+1} + 1}{k+1}, \]

and let

\[ \tilde{S}(n) = \{0^{i_1}1^{i_2}2^{i_3}3^{i_4} \cdots 20^n10^m | \forall i (1 \leq i \leq k+1) [Q+1 \leq n, i] \leq n \}. \]

(Fact 1) There is a subset \( S \) of \( \tilde{S}(n) \) and \( 1 \leq k \leq k+1 \) such that:

(a) \( |S| = (Q+1) \cdot (Q+1) \cdot |Q|^{k+1} + 1 \);

(b) for all \( x \) in \( S \),

\[ x = 0^{i_1}1^{i_2}2^{i_3}3^{i_4} \cdots 20^n10^m \]

For any set \( T \), \( |T| \) denotes the number of elements of \( T \).
\[ z' = 0^{n_1}10^{n_2}20^{n_3}10^{n_4}2 \cdots 20^{n_{k-1}}10^{n_k}1, \]

\[ n_i = n_i', \text{ for } 1 \leq i \leq k + 1 \text{ and } i \neq i'. \]

(Condition c) for all strings in \( S \), there is no head reversal when \( M \) reads \( 0^n10^n \), in the corresponding accepting computations.

(Proof) There are \((n - |Q|)^{k+1}\) strings in \( \overline{S}(n) \). For each of them, there is \( 1 \leq i \leq k + 1 \) such that there is no head reversal when \( M \) reads \( 0^n10^n \). Therefore, there are \( 1 \leq i \leq k + 1 \) and a subset \( S_i \) of \( \overline{S}(n) \) such that \( |S_i| \geq (n - |Q|)^{k+1}/(k + 1) \) and there is no head reversal when \( M \) reads \( 0^n10^n \), for all strings in \( S_i \). There are \((n - |Q|)^k\) possible \( k \) tuples

\[(n_1, n_2, \ldots, n_{k-1}, n_{k+1})\]

with \(|Q| + 1 \leq n_i \leq n \). Hence there is a subset \( S \) such that

\[ |S| \geq \frac{(n - |Q|)^{k+1}}{k + 1} \cdot \frac{1}{(n - |Q|)^{k+1}} = \frac{n - |Q|}{k + 1} \]

that satisfies (b) and (c). It also satisfies (a) because of Eq. (1).

We let \( x = 0^{m_1}10^{m_2} \cdots 20^{m_k}10^{m_{k-1}} \) (\( x = \varepsilon \) if \( i = 1 \)) and \( y = 20^{m_k}10^{m_{k-1}} \cdots 20^{m_2}10^{m_1} \) (\( y = \varepsilon \) if \( i = k + 1 \)). Hence, each string in \( S \) is of the form \( x0^n10^n y \) (with the same \( x \) and \( y \)).

(Fact 2) For all strings in \( S \), during the \(|Q| + 1\) steps after reading \( 0^n1 \), the counter head of \( M \) must move (left or right) at least once. (Note that \(|Q| + 1\) steps include the step that follows reading the 1.)

(Proof) Otherwise, there is \( x0^n10^n y \) in \( S \) such that the counter head of \( M \) does not move \(|Q| + 1\) steps after reading \( 0^n1 \). Using a pumping technique \( 0^n1 = 0^n10^n10^n \), \( n_i = 0(n_i \geq |Q| + 1) \) by the definition of \( S(n) \) and \( M \) also accepts \( x0^n10^n y \). Hence, \( M \) moves for the first time immediately after the input head completes reading \( 10^n \). (Possibly \( n_i = 0 \)). This head movement defines a boundary on the counter tape of \( M \), a crossing sequence at that boundary. While \( M \) reads an input string in \( S(k + 1) \), it crosses the boundary defined above at most \( k + 1 \) times. On the other hand, by (c) of Fact 1, \( M \) crosses this boundary exactly once while reading \( 0^n10^n \). We let this crossing be the \( p \)-th crossing \((1 \leq p \leq k + 1)\) of this boundary. The number of different crossing sequences of \( M \) is at most \(|Q|^{k+1}\), and the number of possible \( 0^n1s \) is \(|Q| + 1 \). Hence by (a) of Fact 1, there are two strings in \( S \):

\[ w = x0^n10^n10^n y, \text{ and} \]

\[ w' = x0^n10^n10^n y \]

with \( m_i = m_i', m_3 \neq m_3' \), with the same crossing sequences at the corresponding boundaries, and the same \( p \). By (c) of Fact 1, \( M \) crosses the corresponding boundary exactly once while reading \( 0^n10^n \) for both \( w \) and \( w' \). By Lemma 1, \( M \) also accepts two mixed versions of \( w \) and \( w' \). One is of the form \( x0^n10^n10^n y \) and one is of the form \( x0^n10^n10^n y \). Both strings are not in \( S(k + 1) \). This is a contradiction.

\[ \text{(Q.E.D.)} \]

4. Conclusions

In this short paper, we show that nearly twice the number of reversals bring out the increase of accepting powers of real-time one counter machines. It is unknown whether or not \( k + 1 \) reversals are more powerful than \( k \) reversals for \( k \geq 1 \).

References
