Uncertainty Relation on Generalized Wigner-Yanase-Dyson Skew Information

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Abstract. We give a trace inequality related to the uncertainty relation of generalized Wigner-Yanase-Dyson skew information which is two parameter's extension of our result in [12].

Key Words: Uncertainty relation, Wigner-Yanase-Dyson skew information

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1 Introduction

Wigner-Yanase skew information

$$\begin{aligned} I_\rho(H) &= \frac{1}{2}Tr\left[\left(i\left[\rho^{1/2}, H\right]\right)^2\right] \\ &= Tr[\rho H^2] - Tr[\rho^{1/2}H\rho^{1/2}H] \end{aligned}$$

was defined in [9]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H. Here we denote the commutator by $[X, Y] = XY - YX$. This quantity was generalized by Dyson

$$\begin{aligned} I_{\rho,\alpha}(H) &= \frac{1}{2}Tr[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])] \\ &= Tr[\rho H^2] - Tr[\rho^\alpha H\rho^{1-\alpha}H], \alpha \in [0, 1] \end{aligned}$$

which is known as the Wigner-Yanase-Dyson skew information. It is famous that the convexity of $I_{\rho,\alpha}(H)$ with respect to ρ was successfully proven by E.H.Lieb in

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[6]. And also this quantity was generalized by Cai and Luo

$$\begin{aligned} & I_{\rho,\alpha,\beta}(H) \\ = \; & \frac{1}{2}Tr[(i[\rho^{\alpha}, H])(i[\rho^{\beta}, H])\rho^{1-\alpha-\beta}] \\ = \; & \frac{1}{2}\{Tr[\rho H^2] + Tr[\rho^{\alpha+\beta}H\rho^{1-\alpha-\beta}H] - Tr[\rho^{\alpha}H\rho^{1-\alpha}H] - Tr[\rho^{\beta}H\rho^{1-\beta}H]\}, \end{aligned}$$

where $\alpha, \beta \geq 0, \alpha+\beta \leq 1$. The convexity of $I_{\rho,\alpha,\beta}(H)$ with respect to ρ was proven by Cai and Luo in [2] under some restrictive condition. From the physical point of view, an observable H is generally considered to be an unbounded opetrator, however in the present paper, unless otherwise stated, we consider $H \in B(\mathcal{H})$ represents the set of all bounded linear operators on the Hilbert space $\mathcal{H}$, as a mathematical interest. We also denote the set of all self-adjoint operators (observables) by $\mathcal{L}_h(\mathcal{H})$ and the set of all density operators (quantum states) by $\mathcal{S}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$. The relation between the Wigner-Yanase skew information and the uncertainty relation was studied in [8]. Moreover the relation between the Wigner-Yanase-Dyson skew information and the uncertainty relation was studied in [5, 10]. In our paper [10] and [12], we defined a generalized skew information and then derived a kind of an uncertainty relations. In the section 2, we discuss various properties of Wigner-Yanase-Dyson skew information. In section 3, we give an uncertainty relation of generalized Wigner-Yanase-Dyson skew information.

2 Trace inequality of Wigner-Yanase-Dyson skew information

We review the relation between the Wigner-Yanase skew information and the uncertainty relation. In quantum mechanical system, the expectation value of an observable H in a quantum state ρ is expressed by $Tr[\rho H]$. It is natural that the variance for a quantum state ρ and an observable H is defined by $V_\rho(H) = Tr[\rho(H - Tr[\rho H]I)^2] = Tr[\rho H^2] - Tr[\rho H]^2$. It is famous that we have

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2 \tag{2.1}$$

for a quantum state ρ and two observables A and B. The further strong results was given by Schrödinger

$$V_\rho(A)V_\rho(B) - |Cov_\rho(A, B)|^2 \geq \frac{1}{4}|Tr[\rho[A, B]]|^2,$$

where the covariance is defined by $Cov_\rho(A, B) = Tr[\rho(A - Tr[\rho A]I)(B - Tr[\rho B]I)]$. However, the uncertainty relation for the Wigner-Yanase skew information failed. (See [8, 5, 10])

$$I_\rho(A)I_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2.$$

Recently, S.Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2}, \tag{2.2}$$

then he derived the uncertainty relation on $U_\rho(H)$ in [7]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|Tr[\rho[A,B]]|^2. \tag{2.3}$$

Note that we have the following relation

$$0 \leq I_\rho(H) \leq U_\rho(H) \leq V_\rho(H). \tag{2.4}$$

The inequality (2.3) is a refinement of the inequality (2.1) in the sense of (2.4). In [12], we studied one-parameter extended inequality for the inequality (2.3).

Definition 2.1 *For $0 \leq \alpha \leq 1$, a quantum state ρ and an observable H, we define the Wigner-Yanase-Dyson skew information*

$$\begin{aligned} I_{\rho,\alpha}(H) &= \frac{1}{2}Tr[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] \\ &= Tr[\rho H_0^2] - Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0] \end{aligned} \tag{2.5}$$

and we also define

$$\begin{aligned} J_{\rho,\alpha}(H) &= \frac{1}{2}Tr[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}] \\ &= Tr[\rho H_0^2] + Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0], \end{aligned} \tag{2.6}$$

where $H_0 = H - Tr[\rho H]I$ and we denote the anti-commutator by $\{X,Y\} = XY + YX$.

Note that we have

$$\frac{1}{2}Tr[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] = \frac{1}{2}Tr[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])]$$

but we have

$$\frac{1}{2}Tr[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}] \neq \frac{1}{2}Tr[\{\rho^\alpha, H\}\{\rho^{1-\alpha}, H\}].$$

Then we have the following inequalities:

$$I_{\rho,\alpha}(H) \leq I_\rho(H) \leq J_\rho(H) \leq J_{\rho,\alpha}(H), \tag{2.7}$$

since we have $Tr[\rho^{1/2}H\rho^{1/2}H] \leq Tr[\rho^{\alpha}H\rho^{1-\alpha}H]$. (See [1, 3] for example.) If we define

$$U_{\rho,\alpha}(H) = \sqrt{V_{\rho}(H)^2 - (V_{\rho}(H) - I_{\rho,\alpha}(H))^2}, \tag{2.8}$$

as a direct generalization of Eq.(2.2), then we have

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_{\rho}(H) \tag{2.9}$$

due to the first inequality of (2.7). We also have

$$U_{\rho,\alpha}(H) = \sqrt{I_{\rho,\alpha}(H)J_{\rho,\alpha}(H)}.$$

From the inequalities (2.4),(2.8),(2.9), our situation is that we have

$$0 \leq I_{\rho,\alpha}(H) \leq I_{\rho}(H) \leq U_{\rho}(H)$$

and

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_{\rho}(H).$$

We gave the following uncertainty relation with respect to $U_{\rho,\alpha}(H)$ as a direct generalization of the inequality (2.3).

Theorem 2.1 ([12]) *For $0 \leq \alpha \leq 1$, a quantum state ρ and observablea A, B,*

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \alpha(1-\alpha)|Tr[\rho[A,B]]|^2. \tag{2.10}$$

Now we define the two parameter extensions of Wigner-Yanase skew information and give an uncertainty relation under some conditions in the next section.

Definition 2.2 *For $\alpha, \beta \geq 0$, a quantum state ρ and an observable H, we define the generalized Wigner-Yanase-Dyson skew information*

$$\begin{aligned} & I_{\rho,\alpha,\beta}(H) \\ = \; & \frac{1}{2}Tr\left[(i[\rho^{\alpha}, H_0])(i[\rho^{\beta}, H_0])\rho^{1-\alpha-\beta}\right] \\ = \; & \frac{1}{2}\{Tr[\rho H_0^2] + Tr[\rho^{\alpha+\beta}H_0\rho^{1-\alpha-\beta}H_0] - Tr[\rho^{\alpha}H_0\rho^{1-\alpha}H_0] - Tr[\rho^{\beta}H_0\rho^{1-\beta}H_0]\} \end{aligned}$$

and we define

$$\begin{aligned} & J_{\rho,\alpha,\beta}(H) \\ = \; & \frac{1}{2}Tr\left[\{\rho^{\alpha}, H_0\}\{\rho^{\beta}, H_0\}\rho^{1-\alpha-\beta}\right] \\ = \; & \frac{1}{2}\{Tr[\rho H_0^2] + Tr[\rho^{\alpha+\beta}H_0\rho^{1-\alpha-\beta}H_0] + Tr[\rho^{\alpha}H_0\rho^{1-\alpha}H_0] + Tr[\rho^{\beta}H_0\rho^{1-\beta}H_0]\}, \end{aligned}$$

where $H_0 = H - Tr[\rho H]I$ *and we denote the anti-commutator by* $\{X, Y\} = XY + YX$*. We remark that* $\alpha + \beta = 1$ *implies* $I_{\rho,\alpha}(H) = I_{\rho,\alpha,1-\alpha}(H)$ *and* $J_{\rho,\alpha}(H) = J_{\rho,\alpha,1-\alpha}(H)$*. We also define*

$$U_{\rho,\alpha,\beta}(H) = \sqrt{I_{\rho,\alpha,\beta}(H) J_{\rho,\alpha,\beta}(H)}.$$

3 Main Theorem

We give the main theorem as follows;

Theorem 3.1 *Let* ρ *be a density operator,* A *and* B *observables and* $\alpha, \beta \geq 0$*. If* $\alpha + \beta \leq \frac{1}{2}$ *or* $\alpha + \beta = 1$*, then the uncertainty relation*

$$U_{\rho,\alpha,\beta}(A) U_{\rho,\alpha,\beta}(B) \geq \alpha\beta |Tr[\rho[A, B]]|^2 \tag{3.1}$$

holds.

We use the several lemmas to prove the theorem 3.1. By spectral decomposition, there exists an orthonormal basis $\{\phi_i,\}_{i=1}^{\infty}$ consisting of eigenvectors of ρ. Let $\{\lambda_i\}_{i=1}^{\infty}$ be the corresponding eigenvalues, where $\sum_{i=1}^{\infty} \lambda_i = 1$ and $\lambda_i > 0$. Thus, ρ has a spectral representation

$$\rho = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|. \tag{3.2}$$

We use the notation $f_\alpha(x, y) = x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha$. Then we have the following lemmas.

Lemma 3.1

$$I_{\rho,\alpha,\beta}(H) = \frac{1}{2} \sum_{i<j} \{\lambda_i + \lambda_j + f_{\alpha+\beta}(\lambda_i, \lambda_j) - f_\alpha(\lambda_i, \lambda_j) - f_\beta(\lambda_i, \lambda_j)\} |\langle\phi_i|H_0|\phi_j\rangle|^2.$$

Proof of Lemma 3.1. By (3.2),

$$\rho H_0^2 = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i| H_0^2.$$

Then

$$Tr[\rho H_0^2] = \sum_i \lambda_i \langle\phi_i|H_0^2|\phi_i\rangle = \sum_i \lambda_i \| H_0 |\phi_i\rangle \|^2. \tag{3.3}$$

Since

$$\rho^\alpha H_0 = \sum_i \lambda_i^\alpha |\phi_i\rangle\langle\phi_i|H_0$$

and

$$\rho^{1-\alpha} H_0 = \sum_i \lambda_i^{1-\alpha} |\phi_i\rangle\langle\phi_i|H_0,$$

we have

$$\rho^\alpha H_0 \rho^{1-\alpha} H_0 = \sum_{i,j} \lambda_i^\alpha \lambda_j^{1-\alpha} |\phi_i\rangle\langle\phi_i|H_0|\phi_j\rangle\langle\phi_j|H_0.$$

Thus

$$\begin{aligned} Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0] &= \sum_{i,j} \lambda_i^\alpha \lambda_j^{1-\alpha} \langle\phi_i|H_0|\phi_j\rangle\langle\phi_j|H_0|\phi_i\rangle \\ &= \sum_{i,j} \lambda_i^\alpha \lambda_j^{1-\alpha} |\langle\phi_i|H_0|\phi_j\rangle|^2. \end{aligned} \tag{3.4}$$

By the similar calculations we have

$$\begin{aligned} Tr[\rho^\beta H_0 \rho^{1-\beta} H_0] &= \sum_{i,j} \lambda_i^\beta \lambda_j^{1-\beta} \langle\phi_i|H_0|\phi_j\rangle\langle\phi_j|H_0|\phi_i\rangle \\ &= \sum_{i,j} \lambda_i^{\alpha+\beta} \lambda_j^{1-\alpha-\beta} |\langle\phi_i|H_0|\phi_j\rangle|^2. \end{aligned} \tag{3.5}$$

$$\begin{aligned} Tr[\rho^{\alpha+\beta} H_0 \rho^{1-\alpha-\beta} H_0] &= \sum_{i,j} \lambda_i^{\alpha+\beta} \lambda_j^{1-\alpha-\beta} \langle\phi_i|H_0|\phi_j\rangle\langle\phi_j|H_0|\phi_i\rangle \\ &= \sum_{i,j} \lambda_i^{\alpha+\beta} \lambda_j^{1-\alpha-\beta} |\langle\phi_i|H_0|\phi_j\rangle|^2. \end{aligned} \tag{3.6}$$

From (2.5), (3.3), (3.4), (3.5), (3.6),

$$\begin{aligned} & I_{\rho,\alpha,\beta}(H) \\ = \; & \frac{1}{2}\sum_{i,j}(\lambda_i + \lambda_i^{\alpha+\beta}\lambda_j^{1-\alpha-\beta} - \lambda_i^\alpha\lambda_j^{1-\alpha} - \lambda_i^\beta\lambda_j^{1-\beta})|\langle\phi_i|H_0|\phi_j\rangle|^2 \\ = \; & \frac{1}{2}\sum_i(\lambda_i + \lambda_i - \lambda_i - \lambda_i)|\langle\phi_i|H_0|\phi_i\rangle|^2 \\ & + \frac{1}{2}\sum_{i<j}(\lambda_i + \lambda_i^{\alpha+\beta}\lambda_j^{1-\alpha-\beta} - \lambda_i^\alpha\lambda_j^{1-\alpha} - \lambda_i^\beta\lambda_j^{1-\beta})|\langle\phi_i|H_0|\phi_j\rangle|^2 \\ & + \frac{1}{2}\sum_{i<j}(\lambda_j + \lambda_j^{\alpha+\beta}\lambda_i^{1-\alpha-\beta} - \lambda_j^\alpha\lambda_i^{1-\alpha} - \lambda_j^\beta\lambda_i^{1-\beta})|\langle\phi_j|H_0|\phi_i\rangle|^2 \\ = \; & \frac{1}{2}\sum_{i<j}(\lambda_i + \lambda_j + f_{\alpha+\beta}(\lambda_i,\lambda_j) - f_\alpha(\lambda_i,\lambda_j) - f_\beta(\lambda_i,\lambda_j))|\langle\phi_i|H_0|\phi_j\rangle|^2. \end{aligned}$$

□

Lemma 3.2

$$J_{\rho,\alpha,\beta}(H) \geq \frac{1}{2}\sum_{i<j}(\lambda_i+\lambda_j+f_{\alpha+\beta}(\lambda_i,\lambda_j)+f_\alpha(\lambda_i,\lambda_j)+f_\beta(\lambda_i,\lambda_j))|\langle\phi_i|H_0|\phi_j\rangle|^2.$$

Proof of Lemma 3.2. By (2.6), (3.3), (3.4), (3.5), (3.6), we have

$$\begin{aligned}
& J_{\rho,\alpha}(H) \\
= & \frac{1}{2}\sum_{i,j}(\lambda_i+\lambda_i^{\alpha+\beta}\lambda_j^{1-\alpha-\beta}+\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{\beta}\lambda_j^{1-\beta})|\langle\phi_i|H_0|\phi_j\rangle|^2 \\
= & \frac{1}{2}\sum_{i}(\lambda_i+\lambda_i+\lambda_i+\lambda_i)|\langle\phi_i|H_0|\phi_i\rangle|^2 \\
& +\frac{1}{2}\sum_{i<j}(\lambda_i+\lambda_i^{\alpha+\beta}\lambda_j^{1-\alpha-\beta}+\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{\beta}\lambda_j^{1-\beta})|\langle\phi_i|H_0|\phi_j\rangle|^2 \\
& +\frac{1}{2}\sum_{i<j}(\lambda_j+\lambda_j^{\alpha+\beta}\lambda_i^{1-\alpha-\beta}+\lambda_j^{\alpha}\lambda_i^{1-\alpha}+\lambda_j^{\beta}\lambda_i^{1-\beta})|\langle\phi_j|H_0|\phi_i\rangle|^2 \\
= & 2\sum_{i}\lambda_i|\langle\phi_i|H_0|\phi_i\rangle|^2 \\
& +\frac{1}{2}\sum_{i<j}(\lambda_i+\lambda_j+f_{\alpha+\beta}(\lambda_i,\lambda_j)+f_\alpha(\lambda_i,\lambda_j)+f_\beta(\lambda_i,\lambda_j)|\langle\phi_i|H_0|\phi_j\rangle|^2 \\
\geq & \frac{1}{2}\sum_{i<j}(\lambda_i+\lambda_j+f_{\alpha+\beta}(\lambda_i,\lambda_j)+f_\alpha(\lambda_i,\lambda_j)+f_\beta(\lambda_i,\lambda_j)|\langle\phi_i|H_0|\phi_j\rangle|^2.
\end{aligned}$$

□

Lemma 3.3 *For any $t>0$ and $\alpha,\beta\geq 0, \alpha+\beta\geq 1$ or $\alpha+\beta\leq\frac{1}{2}$, the following inequality holds;*

$$(t^{1-\alpha-\beta}+1)^2(t^{2\alpha}-1)(t^{2\beta}-1)\geq 16\alpha\beta(t-1)^2. \tag{3.7}$$

Proof of Lemma 3.3. It is sufficient to prove (3.7) for $t\geq 1$ and $\alpha,\beta\geq 0, \alpha+\beta\geq 1$ or $\alpha+\beta\leq\frac{1}{2}$. By Lemma 3.3 in [12] we have for $0\leq p\leq 1$ and $s\geq 1$,

$$(1-2p)^2(s-1)^2-(s^p-s^{1-p})^2\geq 0.$$

Then we can rewrite as follows;

$$(s^{2p}-1)(s^{2(1-p)}-1) \geq 4p(1-p)(s-1)^2.$$

We assume that $\alpha, \beta \geq 0$. We put $p = \alpha/(\alpha+\beta)$ and $s^{1/(\alpha+\beta)} = t$. Then

$$(t^{2\alpha}-1)(t^{2\beta}-1) \geq \frac{4\alpha\beta}{(\alpha+\beta)^2}(t^{\alpha+\beta}-1)^2.$$

Then we have

$$(t^{1-\alpha-\beta}+1)^2(t^{2\alpha}-1)(t^{2\beta}-1) \geq \frac{4\alpha\beta}{(\alpha+\beta)^2}(t^{1-\alpha-\beta}+1)^2(t^{\alpha+\beta}-1)^2. \tag{3.8}$$

In order to have the aimed inequality, we have to show that

$$(t^{1-\alpha-\beta}+1)^2(t^{\alpha+\beta}-1)^2 \geq 4(\alpha+\beta)^2(t-1)^2.$$

It is sufficient to prove the following inequality

$$(t^{1-\alpha-\beta}+1)(t^{\alpha+\beta}-1) \geq 2(\alpha+\beta)(t-1)$$

for $t \geq 1$ and $\alpha, \beta \geq 0, \alpha+\beta \geq 1$ or $\alpha+\beta \leq \frac{1}{2}$. We put $\alpha+\beta = k$ and $f(t) = (t^{1-k}+1)(t^k-1) - 2k(t-1)$. Then

$$\begin{aligned} f'(t) &= (1-k)t^{-k}(t^k-1) + k(t^{1-k}+1)t^{k-1} - 2k \\ &= (1-k)(1-t^{-k}) + k(1+t^{k-1}) - 2k. \end{aligned}$$

and

$$\begin{aligned} f''(t) &= (1-k)kt^{-k-1} + k(k-1)t^{k-2} \\ &= k(k-1)(t^{k-2} - t^{-k-1}). \end{aligned}$$

When $k = \alpha+\beta \geq 1$ or $k = \alpha+\beta \leq \frac{1}{2}$, it is easy to show that $f''(t) \geq 0$ for $t \geq 1$. Since $f'(1) = 0$, we have $f'(t) \geq 0$ for $t \geq 1$. And since $f(1) = 0$, we have $f(t) \geq 0$ for $t \geq 1$. Hence we have for $\alpha+\beta \geq 1$ or $\alpha+\beta \leq \frac{1}{2}$,

$$(t^{1-\alpha-\beta}+1)(t^{\alpha+\beta}-1) \geq 2(\alpha+\beta)(t-1).$$

It follows from (3.8) that we get

$$(t^{1-\alpha-\beta}+1)^2(t^{2\alpha}-1)(t^{2\beta}-1) \geq 16\alpha\beta(t-1)^2.$$

□

Proof of Theorem 3.1. Since

$$\begin{aligned} & (t^{1-\alpha-\beta}+1)^2(t^{2\alpha}-1)(t^{2\beta}-1) \\ = \; & (t+1+t^{\alpha+\beta}+t^{1-\alpha-\beta})^2-(t^{\alpha}+t^{1-\alpha}+t^{\beta}+t^{1-\beta})^2, \end{aligned}$$

we put $t=\frac{\lambda_i}{\lambda_j}$ in (3.7). Then we have

$$\left\{\frac{\lambda_i}{\lambda_j}+1+\left(\frac{\lambda_i}{\lambda_j}\right)^{\alpha+\beta}+\left(\frac{\lambda_i}{\lambda_j}\right)^{1-\alpha-\beta}\right\}^2-\left\{\left(\frac{\lambda_i}{\lambda_j}\right)^{\alpha}+\left(\frac{\lambda_i}{\lambda_j}\right)^{1-\alpha}+\left(\frac{\lambda_i}{\lambda_j}\right)^{\beta}+\left(\frac{\lambda_i}{\lambda_j}\right)^{1-\beta}\right\}^2$$

$$\geq 16\alpha\beta\left(\frac{\lambda_i}{\lambda_j}-1\right)^2.$$

Then we have

$$\begin{aligned} & \{\lambda_i+\lambda_j+f_{\alpha+\beta}(\lambda_i,\lambda_j)-f_{\alpha}(\lambda_i,\lambda_j)-f_{\beta}(\lambda_i,\lambda_j)\} \\ & \times\{\lambda_i+\lambda_j+f_{\alpha+\beta}(\lambda_i,\lambda_j)+f_{\alpha}(\lambda_i,\lambda_j)+f_{\beta}(\lambda_i,\lambda_j)\} \\ = \; & (\lambda_i+\lambda_j+f_{\alpha+\beta}(\lambda_i,\lambda_j))^2-(f_{\alpha}(\lambda_i,\lambda_j)+f_{\beta}(\lambda_i,\lambda_j))^2 \\ \geq \; & 16\alpha\beta(\lambda_i-\lambda_j)^2. \end{aligned} \tag{3.9}$$

Since

$$\begin{aligned} Tr[\rho[A,B]] &= Tr[\rho[A_0,B_0]] \\ &= 2iImTr[\rho A_0B_0] \\ &= 2i\sum_{i<j}(\lambda_i-\lambda_j)Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle, \end{aligned}$$

$$\begin{aligned} |Tr[\rho[A,B]]| &= 2|\sum_{i<j}(\lambda_i-\lambda_j)Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle| \\ &\leq 2\sum_{i<j}|\lambda_i-\lambda_j||Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle|. \end{aligned}$$

Then we have

$$|Tr[\rho[A,B]]|^2\leq 4\left\{\sum_{i<j}|\lambda_i-\lambda_j||Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle|\right\}^2.$$

By (3.9) and Schwarz inequality,

$$\alpha\beta|Tr[\rho[A,B]]|^2$$

$$
\begin{aligned}
&\leq 4\alpha\beta\left\{\sum_{i<j}|\lambda_i-\lambda_j||Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle|\right\}^2\\
&= \frac{1}{4}\left\{\sum_{i<j}4\sqrt{\alpha\beta}|\lambda_i-\lambda_j||Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle|\right\}^2\\
&\leq \frac{1}{4}\left\{\sum_{i<j}4\sqrt{\alpha\beta}|\lambda_i-\lambda_j||\langle\phi_i|A_0|\phi_j\rangle||\langle\phi_j|B_0|\phi_i\rangle|\right\}^2\\
&\leq \frac{1}{4}\left\{\sum_{i<j}\{K^2-L^2\}^{1/2}|\langle\phi_i|A_0|\phi_j\rangle||\langle\phi_j|B_0|\phi_i\rangle|\right\}^2\\
&\leq \frac{1}{2}\sum_{i<j}(K-L)|\langle\phi_i|A_0|\phi_j\rangle|^2\times\frac{1}{2}\sum_{i<j}(K+L)|\langle\phi_i|B_0|\phi_j\rangle|^2,
\end{aligned}
$$

where $K=\lambda_i+\lambda_j+f_{\alpha+\beta}(\lambda_i,\lambda_j)$ and $L=f_\alpha(\lambda_i,\lambda_j)+f_\beta(\lambda_i,\lambda_j)$. Then we have

$$I_{\rho,\alpha,\beta}(A)J_{\rho,\alpha,\beta}(B)\geq\alpha\beta|Tr[\rho[A,B]]|^2.$$

We also have

$$I_{\rho,\alpha,\beta}(B)J_{\rho,\alpha,\beta}(A)\geq\alpha\beta|Tr[\rho[A,B]]|^2.$$

Hence we have the final result (3.1). □

Remark 3.1 *We remark that (2.10) is derived by putting $\beta=1-\alpha$ in (3.1). Then Theorem 3.1 is a generalization of Theorem 2.1 given in [12]. Moreover, considering the proof, if the dimension is finite and the density operator ρ is invertible, then (3.1) holds even if $\alpha+\beta\geq 1$.*

Remark 3.2 *When $\alpha,\beta\geq 0$ and $\frac{1}{2}<\alpha+\beta<1$, we can show an example which Theorem 3.1 does not hold as follows; Let*

$$\rho=\begin{pmatrix}\frac{3}{4} & 0\\ 0 & \frac{1}{4}\end{pmatrix}, A=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}, B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \alpha=\frac{1}{2}, \beta=\frac{1}{4}.$$

Then we have

$$U_{\rho,\alpha,\beta}(A)U_{\rho,\alpha,\beta}(B)=0.00448729,$$
$$\alpha\beta\left|Tr[\rho[A,B]]\right|^2=0.125.$$

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