Some Structural Properties of Product Automata

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Abstract

Some properties of product and self-product automata are analyzed by the product of state transition matrices.

It can be shown that the self-product automata of cyclic type automata with \(2m + 1\) (\(m \geq 1\)) states are again cyclic, and the relationships between self-product automata are made clear. As the most important result, we show that a set of self-product automata with some states forms a group with the product operation of automata. Moreover, concerning the number of states of automata, the necessary and sufficient condition to form the group can be derived. Furthermore, it can be shown that the sequence of self-product automata with \(2^m\) (\(m \geq 1\)) states converges to the unity type automata.

1. Introduction

The automata model, whose aspects vary with the lapse of time, can be found in the theories of nonhomogeneous markov chains and L-systems. In a recent paper, A. Paz\(^{1,2,3}\), J. Hajnal\(^4\), and J. Wolfowitz\(^5\) gave some properties of stochastic matrices and discussed ergodicity of the nonhomogeneous markov chains, and H. Jürgensen\(^6\) investigated some limiting properties of the system introducing the probabilistic factors into L-systems. From the automata-theoretical point of view\(^7\)\(^8\), it will be interesting to study the transition structures of those time-variant systems.

In this paper, we introduce a product automaton defined by the product of the matrices which represent the state transitions, and investigate the transition structures of the product automata. In order to simplify the problem, we deal with a cyclic type of DAOA, where “DAOA” denotes the deterministic autonomous outputless automaton. Furthermore, we introduce a new concept of self-product automata, and derive some algebraic properties of them.

In Section 2, we give some definitions and notations concerning DAOA’s, and give several fundamental propositions derived from those definitions. In Section 3, we investigate the transition structures of product automata, and in Section 4, we investigate some algebraic properties of a sequence of self-product automata.

2. Preliminaries

In this section, we give some definitions concerning autonomous outputless automata, and give some fundamental propositions derived from those definitions.

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Notation 2.1

(2.1.1) Let $A$ be a square matrix of order $n$ for some $n \geq 1$. Then $A_{ij}$ denotes the 
ij entry of $A$ ($1 \leq i, j \leq n$).

(2.1.2) For any set $U$, $|U|$ denotes the number of elements of $U$.

Definition 2.1. An autonomous outputless automaton (AOA) is a couple $S = [Z, A]$, where "$Z$" is a finite set of states, and "$A$" is a square matrix of order $|Z|$ whose 
each entry $A_{ij}$ ($1 \leq i, j \leq |Z|$) is 1 or 0.

Definition 2.2. Let $S = [Z, A]$ be an AOA. The directed graph $G = [V, E]$ is 
called the state transition graph of $S$, where $V = Z$, $E = \{(z_i, z_j) \in Z \times Z | A_{ij} = 1\}$ and 
$(z_i, z_j)$ denotes the edge directed from $z_i$ to $z_j$.

Let $Z = \{z_1, z_2, \ldots, z_n\}$ ($n = |Z|$). For each $i, j$ ($1 \leq i, j \leq n$), "$A_{ij} = 1$" means that 
when the present state of $S$ is $z_i$, one of the next states of $S$ is $z_j$. Also, "autonomous" 
means that the behaviour of $S$ is independent of the input signal.

Definition 2.3. Let $A$ be a square matrix of order $n$ for some $n \geq 1$. A is called 
"deterministic" if $\sum_{j=1}^{n} A_{ij} = 1$ for all $i \in \{1, 2, \ldots, n\}$.

Definition 2.4. An AOA $S = [Z, A]$ is called "deterministic" if $A$ is deterministic. We 
denote a deterministic AOA by "DAOA".

For a deterministic matrix, a proposition is given as follows.

Proposition 2.1. If $P$ and $Q$ are deterministic matrices, then the product matrix $PQ$ is also a deterministic matrix.

The proof of this proposition and subsequent propositions are omitted, because they are obvious.

Definition 2.5. $S^o(Z) \triangleleft [Z, I]$ is called a "unity" DAOA with the state set $Z$, 
where $I$ is an unit matrix of order $|Z|$.

A DAOA can be considered as a kind of directed graph. Therefore, this paper 
may play a special role of connecting automata theory with graph theory.

Definition 2.6. Let $S_1 = [Z_1, A_1]$ and $S_2 = [Z_2, A_2]$ be DAOA's. Then $S_1 \triangleleft S_2$ if 
and only if $Z_1 = Z_2$ and $A_1 = A_2$.

Definition 2.7. Let $S_1 = [Z, A_1], S_2 = [Z, A_2], \ldots, S_n = [Z, A_n]$ ($n \geq 2$) be 
DAOA's with the same set of states, $Z$. Then, the DAOA $[Z, A_1 A_2 \cdots A_n]$ is called the 
product automaton of $S_1, S_2, \ldots, S_n$, and is denoted by "$S_1 \triangleleft S_2 \cdots S_n$".

Definition 2.8. Let $S = [Z, A]$ be a DAOA. Then, $S^m$ ($m \geq 1$) is called as self- 
product automaton (where $S^m = S \cdots S \triangleleft [Z, A^m]$).
Definition 2.9. Let \( S = [Z, A] \) be a DAOA, and \( Z = \{ z_1, z_2, \ldots, z_n \} \) (\( n = |Z| \)).

(2.9.1) For any two states \( z_i, z_j \in Z \), we call that the state \( z_j \) is accessible from the state \( z_i \) if there exists a natural number \( m \) such that \( A_{i,j}^m = 1 \).

(2.9.2) \( S \) is called "cyclic" or "connected" if, for any two states \( z_i, z_j \in Z \), \( z_j \) is accessible from \( z_i \) and vice versa. We denote a cyclic DAOA by "CDAOA". And, for each CDAOA \( S = [Z, A] \), \( A \) is called a cyclic matrix.

We note that, for any CDAOA \( S = [Z, A] \), there exists exactly one "1" on each row and each column of \( A \), respectively.

Definition 2.10.

(2.10.1) Let \( Z \) be the set of \( n \) states \( z_1, z_2, \ldots, z_n \). Then, a mapping \( \tau : Z \to Z \) satisfying the following condition (C) is called a state renaming function for \( Z \).

\[
(C) \quad \forall i, j \in Z, \quad [i \to j] \Rightarrow \tau(z_i) = \tau(z_j).
\]

A state renaming function \( \tau \) for \( Z \) is simply denoted by "SRF(Z)" or "SRF(Z) \( \tau \)."

(2.10.2) Let \( S = [Z, A] \) be a DAOA and \( \tau \) be a SRF(Z). Then we define \( \tau(A) \) to be the deterministic matrix of order \( |Z| \) as follows. For each \( i, j \) (\( 1 \leq i, j \leq |Z| \)) \( \tau(A)_{ij} = 1 \) if and only if \( z_i = \tau(z_j) \) and \( z_j = \tau(z_i) \) and \( A_{p,q} = 1 \).

Definition 2.11. For any \( n \geq 1 \), the square matrix of order \( n \)

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]
is called "canonical".

Proposition 2.2.

(2.2.1) If \( S = [Z, A] \) is a cyclic (non-cyclic) DAOA, then \( S' = [Z, \tau(A)] \) is again a cyclic (non-cyclic) DAOA for any SRF(Z) \( \tau \).

(2.2.2) For any CDAOA \( S = [Z, A] \), there exists a SRF(Z) \( \tau \) such that \( \tau(A) \) is canonical.

(2.2.3) For any DAOA \( S = [Z, A] \) and any SRF(Z) \( \tau \), there exists a SRF(Z) \( \tau' \) such that \( \tau'(\tau(A)) = A \).

(2.2.4) Let \( S_1 = [Z, A_1] \) and \( S_2 = [Z, A_2] \) be any DAOA's. Then, for any SRF(Z) \( \tau \), \( \tau(A_1A_2) = \tau(A_1)\tau(A_2) \).

(2.2.5) Let \( S = [Z, A] \) be a DAOA. Then, for any SRF(Z)'s \( \tau' \) and \( \tau^* \), there exists a SRF(Z) \( \tau \) such that \( \tau'(\tau^*(A)) = \tau(A) \).

(2.2.6) For any \( S^*(Z) = [Z, I] \) and any SRF(Z) \( \tau, \tau(I) = 1 \).

Definition 2.12. Let \( A \) and \( B \) be two deterministic matrices of order \( r \geq 1 \) and \( s \geq 1 \), respectively. Then the matrix
\[
A \otimes B \triangleq \begin{pmatrix}
A & O \\
O & B
\end{pmatrix}
\]
of order \(r+s\) is called the direct sum of \(A\) and \(B\).

**Definition 2.13.** Let \(R\) denote a form
\[
R = \begin{pmatrix}
i_1 & i_2 & \cdots & i_s \\
j_1 & j_2 & \cdots & j_s
\end{pmatrix} \quad (n \geq 1)
\]
where, (1) for each \(p(1 \leq p \leq n)\), \(i_p\) and \(j_p\) are natural number, and (2) for each \(p\), \(p(1 \leq p < q \leq n)\) \(i_p \neq i_q\). If \(\{j_1, j_2, \ldots, j_n\} \subseteq \{i_1, i_2, \ldots, i_n\}\), then \(R\) is called “pseudo permutation”. Especially, if \(\{j_1, j_2, \ldots, j_n\} = \{i_1, i_2, \ldots, i_n\}\), then \(R\) is called “permutation”.

**Definition 2.14.** Let \(C\) and \(D\) are pseudo permutations as follows.

\[
C = \begin{pmatrix}
i_1 & i_2 & \cdots & i_s \\
j_1 & j_2 & \cdots & j_s
\end{pmatrix}, \quad D = \begin{pmatrix}
r_1 & r_2 & \cdots & r_s \\
s_1 & s_2 & \cdots & s_s
\end{pmatrix} \quad (n \geq 1)
\]
where \(\{i_1, i_2, \ldots, i_n\} = \{r_1, r_2, \ldots, r_n\}\). Then

\[
CD \triangleq \begin{pmatrix}
i_1 & i_2 & \cdots & i_s \\
u_1 & u_2 & \cdots & u_s
\end{pmatrix},
\]
where, for each \(i(1 \leq i \leq n)\), if \(j_i = r_k(1 \leq k \leq n)\), then \(u_i = s_k\).

**Convention 2.1** Let \(C\) and \(D\) be the pseudo permutations described in the Definition 2.13. Then, we identify \(D\) with \(C\), if \(D\) is obtained from \(C\) by rearranging each column of \(C\), appropriately. For example.

\[
\begin{pmatrix}1 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix}1 & 3 & 5 & 2 & 4 \end{pmatrix} = \begin{pmatrix}3 & 2 & 1 & 2 & 5 \end{pmatrix}
\]

**Definition 2.15.** A permutation
\[
\begin{pmatrix}
i_1 & i_2 & \cdots & i_s \\
i_2 & i_3 & \cdots & i_1
\end{pmatrix}
\]
is called “cyclic”, and is denoted by \((i_1i_2 \ldots i_n)\).

**Notation 2.2** Let \(A_1, A_2, \ldots,\) and \(A_k(k \geq 1)\) be cyclic permutations such that
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(1) \( A_i = (j_{i1}^1 \cdots j_{ik}^i) \) (\( n_i \geq 1, 1 \leq i \leq k \)), and

(2) \( \{j_p^1, \ldots, j_p^i\} \cap \{j_q^1, \ldots, j_q^i\} = \emptyset \) for each \( p, q \) (1 \( \leq p < q \leq k \)).

(\( \emptyset \) denotes the null set.)

Then, \( A_1 \odot A_2 \odot \cdots \odot A_k \) denotes the permutation

\[
\begin{pmatrix}
  j_{11}^1 & j_{12}^1 & \cdots & j_{1k}^1 & j_{21}^1 & j_{22}^1 & \cdots & j_{2k}^1 & \cdots & j_{ik}^1 \\
  j_{11}^2 & j_{12}^2 & \cdots & j_{1k}^2 & j_{21}^2 & j_{22}^2 & \cdots & j_{2k}^2 & \cdots & j_{ik}^2
\end{pmatrix}
\]

From convention 2.1, note that \( A_1 \odot A_2 = A_2 \odot A_1 \).

**Definition 2.16.** For any deterministic matrix \( A \) of order \( n \geq 1 \), \( \varphi(A) \) is defined as follows.

\[
\varphi(A) \equiv \begin{pmatrix}
  1 & 2 & \cdots & i & \cdots & n \\
  j_1 & j_2 & \cdots & j_i & \cdots & j_n
\end{pmatrix}, \quad j_1, j_2, \ldots, j_i \in \{1, 2, \ldots, n\}
\]

where, for each \( i \) (1 \( \leq i \leq n \)), the \( ij_i \) entry of \( A \) is "1". (Note that there exists exactly one "1" on each row of the deterministic matrix.)

**Proposition 2.3.**

(2.3.1) For any deterministic matrices \( A_1 \) and \( A_2 \) of order \( n \geq 1 \), \( \varphi(A_1) \varphi(A_2) = \varphi(A_1 \odot A_2) \).

(2.3.2) For any cyclic matrix \( A \) of order \( n \geq 1 \) and any SRF(Z) \( \tau \), if \( \varphi(\tau(A)) = \varphi(\tau)^n = \varphi(1) \), then \( \varphi(A^n) = \varphi(1) \).

**Example 1.** Let \( A_1 \) and \( A_2 \) be two deterministic matrices as follows.

\[
A_1 = \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
  1 & 0 & 0 & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 1 & 0
\end{pmatrix}
\]

Then

\[
\varphi(A_1) = \begin{pmatrix}
  1 & 2 & 3 & 4 & 5 \\
  2 & 3 & 5 & 4 & 1
\end{pmatrix}, \quad \varphi(A_2) = \begin{pmatrix}
  1 & 2 & 3 & 4 & 5 \\
  3 & 2 & 1 & 5 & 4
\end{pmatrix}, \quad \text{and}
\]

\[
\varphi(A_1) \varphi(A_2) = \begin{pmatrix}
  1 & 2 & 3 & 4 & 5 \\
  2 & 1 & 4 & 5 & 3
\end{pmatrix} = (1 2) \odot (3 4 5)
\]

**Notation 2.3.**

(2.3.1) The set of all DAOA's (CDAO's) is denoted by \( \& \).
(2.3.2) $\&^{(\epsilon)} \cup \&^{(o)}$ U $\&^{(o)}$, where $\&^{(\epsilon)}$ ($\&^{(o)}$) is the sets of all DAOA's with an even (odd) number of states.

(2.3.3) $\&^{(\epsilon)} \cup \&^{(o)} \cup \&^{(o)}$, where $\&^{(\epsilon)}$ ($\&^{(o)}$) is the sets of all CDAOA's with an even (odd) number of states.

(2.3.4) $\&^{(\epsilon)} \cup \&^{(i)} \cup \&^{(i)}$, $\&^{(o)} \cup \&^{(i)} \cup \&^{(i)}$ ...

$\&^{(\epsilon)} \cup \&^{(i)} \cup \&^{(i)}$, $\&^{(o)} \cup \&^{(i)} \cup \&^{(i)}$ ...

where, for any natural number i, $\&^{[i]}$ ($\&^{[i]}$) is the set of all DAOA's (CDAOA's) with just i states ($\&^{[i]} \neq \emptyset$).

3. The Transition Structures of Product Automata

In this section, we investigate the transition structures of product automata. First, we give a lemma whose proof is obvious from Definition 2.9.2.

Lemma 3.1. Let $S = [Z, A]$ be in $\&^{[n]}$ ($n \geq 1$) and let

$$\varphi (A) = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_s \end{pmatrix}.$$ 

If $i_r = i_s$ for some $r, s \in \{1, 2, \ldots, n\}$ ($r \neq s$), then $S$ is not in $\&^{[n]}$

Theorem 3.1. Let $S_1 = [Z, A_1], S_2 = [Z, A_2]$ be DAOA's, and let

$$\varphi (A_1) = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_s \end{pmatrix}, \quad \varphi (A_2) = \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_s \end{pmatrix}.$$ 

Then, if $i_r = j_l$ or $j_p = j_q$ for some $r, s, p, q \in \{1, 2, \ldots, n\}$ ($r \neq s, p \neq q$), then the product automaton $S_1 S_2 = S_3 = [Z, A_3]$ is not in $\&^{[n]}$

Proof. Suppose that $i_r = i_s$. Then

$$\varphi (A_1) = \varphi (A_1 A_2) = \varphi (A_1) \varphi (A_2)$$

$$= \begin{pmatrix} 1 & \cdots & r & \cdots & s & \cdots & n \\ i_1 & \cdots & i_r & \cdots & i_s \end{pmatrix} \begin{pmatrix} 1 & \cdots & i_r & \cdots & n \\ j_1 & \cdots & j_r \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \cdots & r & \cdots & s & \cdots & n \\ k_s & \cdots & j_1 & \cdots & j_i \end{pmatrix}$$

It follows from Lemma 3.1 that $S_3$ is not in $\&^{[n]}$. The similar argument is used for the case of $j_p = j_q$. Q.E.D.

The following theorem implies that the self-product automata of cyclic DAOA's with an old number of states are again cyclic or unity.
Theorem 3.2. Let $S = [Z, A] \in \&_{i}^{[2n-1]}$ $(m \geq 1)$.

(1) $S^n = S^\circ (Z)$, if $n = r(2m-1)$ for some natural number $r \geq 1$,

(2) $S^n \in \&_{i}^{[2n-1]}$, otherwise.

Proof. (1) Let $S = [Z, A] \in \&_{i}^{[2n-1]}$. Then, for a SRE$(Z)$ $\tau$, the matrix $A$ can be transformed into the canonical matrix $\tau (A)$ of order $2m-1$ (from Proposition 2.2.2), and $\varphi (\tau (A))$ can be denoted as follows.

$$
\varphi (\tau (A)) = \begin{pmatrix}
1 & 2 \cdots 2m-2 & 2m-1 \\
2 & 3 \cdots 2m-1 & 1
\end{pmatrix}
$$

Easily, we have

$$
\varphi (\tau (A))^r \in \&_{i}^{[2n-1]} = \begin{pmatrix}
1 & 2 \cdots 2m-1 \\
2 & 2 \cdots 2m-1
\end{pmatrix},
$$

for any natural number $r \geq 1$. Since $\varphi (\tau (A))^r \in \&_{i}^{[2n-1]} = \varphi (\tau (A)^r \in \&_{i}^{[2n-1]})$ (from Proposition 2.2.4), it follows that

$$
\varphi (\tau (A))^r \in \&_{i}^{[2n-1]} = \begin{pmatrix}
1 & 2 \cdots 2m-1 \\
2 & 2 \cdots 2m-1
\end{pmatrix}
$$

Therefore $\tau (A)^r \in \&_{i}^{[2n-1]}$ is the unit matrix of order $2m-1$. From Propositions 2.2.4 and 2.2.6, it follows that $A^r \in \&_{i}^{[2n-1]}$ is the unit matrix of order $2m-1$, which implies that $S^r \in \&_{i}^{[2n-1]} = S^\circ (Z)$.

(2) To prove (2), from part (1) of the theorem, it suffices to show that, for each $n$ ($1 \leq n \leq 2m-2$), $S^n$ is cyclic, Let $\tau (A)$ be the canonical matrix as the mentioned above. Then

$$
\varphi (\tau (A)^n) = \varphi (\tau (A))^n
$$

From the state transition graph of $S^n = [Z, \tau (A)^n]$, it is easily shown that $S^n$ is cyclic. Therefore, from Propositions 2.2.1 and 2.2.3, it is shown that $S^n = [Z, A^n]$ is cyclic. Q.E.D.

Corollary 3.1. Let $S \in \&_{i}^{[2n-1]}$ $(m \geq 1)$. Then, for each $n \geq 1$, $S^n$ is in $\&_{i}^{[2n-1]}$.

When the number of states is even, the state transition structures of product automata differ from the odd case, as follows.

Theorem 3.3. Let $S = [Z, A] \in \&_{i}^{[2n]}$ $(m \geq 1)$, and let $S^2 = [Z, A^2]$. Then, there exists a SRF$(Z)$ $\tau$ such that $\tau (A^2) = A_1 \oplus A_2$, for some cyclic matrices $A_1$ and $A_2$ of order $m$. 
Proof. For each \( S = [Z, A] \subseteq \mathfrak{d}_3 \), there exists a SRF \( \tau' \) such that the matrix \( \tau'(A) \) (of order 2m) is canonical, and

\[
\tau'(A) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

From the state transition graph of \( S' = [Z, \tau'(A)^2] \), it is easily shown that there exists a SRF \( \tau'' \) such that \( \tau''(\tau'(A)^2) = A_1 \oplus A_2 \), for some cyclic matrices \( A_1 \) and \( A_2 \) of order m. Therefore, from Propositions 2.2.3 and 2.2.4, it follows that, for some SRF \( \tau \), \( \tau(A^2) = A_1 \oplus A_2 \). This completes the proof of the theorem. Q.E.D.

**Theorem 3.4.** For each \( S_1 \subseteq \mathfrak{d}^{[m]}_3 \) \((m \geq 1)\), there exists some \( S_2 \) in \( \mathfrak{d}^{[m]}_3 \) such that \( S_2 \not\subseteq \mathfrak{d}^{[m]}_3 \) and \( S_1 S_2 \subseteq \mathfrak{d}^{[m]}_3 \).

Proof. Let \( S_1 = [Z, A_1] \subseteq \mathfrak{d}^{[m]}_3 \). From Proposition 2.2.2, there exists a SRF \( \tau \) such that \( \tau(A_1) \) is the canonical matrix of order 2m.

Now we consider a non-cyclic matrix \( A \) of order 2m as follows.

\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Then

\[
\tau(A_1)A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

From the state transition graph of \( S' = [Z, \tau(A_1)A] \), it is easily shown that \( S' \) is cyclic. From Proposition 2.2.3, there exists a SRF \( \tau' \) such that \( \tau'(\tau(A_1)) = A_1 \), and thus \( \tau'(\tau(A_1)A) = \tau'(\tau(A_1))\tau'(A) = A_1 \tau'(A) \).

Noting that \( \tau(A_1)A \) is cyclic, from Proposition 2.2.1, it follows that \( A_1 \tau'(A) \) is cyclic. Furthermore, since \( A \) is non-cyclic, \( \tau'(A) \) is also non-cyclic (from Proposition 2.2.1). Thus \( S_2 = [Z, \tau'(A)] \) is not cyclic, but \( S_2 = S_1 S_2 = [Z, A_1 \tau'(A)] \) is cyclic. Q.E.D.
The next theorem implies that the product automaton $S_1S_2$ with an odd number of states is not always cyclic, even if both $S_1$ and $S_2$ are cyclic.

**Theorem 3.5.** There exists $S_1$, $S_2 \subseteq \&_\epsilon \cdot (S_1 \times S_2)$ such that $S_1S_2 \subseteq \&_\epsilon \cdot (S_1 \times S_2)$

**Proof.** Let $S_1 = [Z, A_1]$, $S_2 = [Z, A_2] \subseteq \&_\epsilon \cdot (S_1 \times S_2)$, where

$$
A_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},
A_2 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}.
$$

Then

$$
A_1 = A_1 A_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

It is obvious that $S_2 = [Z, A_2]$ is not in $\&_\epsilon \cdot (S_1 \times S_2)$ (See Fig.1).

![State transition graphs of $S_1$, $S_1^c$ and $S_2$ in Theorem 3.5](image)

**4. Self-product Automata**

In Section 3, we considered the transition structures of product automata of DAOA's. In this section, we investigate some algebraic properties concerning an infinite sequence of self-product automata.

**Notation 4.1.** For some natural number $n$ and $m$, $[n]_m$ denotes the remainder when $n$ is divided by $m$.

**Notation 4.2.**

(4.2.1) Let $S$ be a DAOA. Then $S^*$ denotes the infinite sequence of self-product automata $S$, $S^2$, $S^3$, ... , $S^n$, ... .

(4.2.2) For each DAOA $S = [Z, A]$, $\Omega(S) \subseteq \{S^*(Z) \cup \{S^n \mid n \geq 0\}.$

**Definition 4.1.** Let $S = [Z, A]$ be a DAOA. The least natural number $n$ such that
A^{2n}=A is called a "cycle" of S*, and is denoted by C(S*).

**Proposition 4.1.** Let S=[Z, A]∈&^{[i<n]} (n≥1). If C(S*)=n, then

(4.1.1) \( S^{m-n}=S \), for each natural number, m,
(4.1.2) For any SRF(Z) τ, let \( S_τ=[Z, τ(A)] \), then C(S*)=C(S_τ).

In a sense, the concept of an infinite sequence of self-product automata has a physical and biological image such that the present aspects of a system depends upon its past aspects.

In the following theorem, it is shown that a set \( Ω(S) \) for a CDAOA S consists a group with the product operation of automata.

First, we give the wellknown lemma, as follows.

**Lemma 4.1.** The order of a cyclic group generated from a cyclic permutation (1 2 \( \ldots \) n) is n.

**Theorem 4.1.** If \( S∈&^{[i≤2m+1]} \) and C(S*)=2m, then the set \( S(S) \) forms a group with the product operation of automata.

Proof. Let S=[Z, A]∈&^{[i≤2m+1]} and let a set U be

\[ U = \{ S^i (Z) \} \cup \{ S, S^2, \ldots, S^{2m-1} \} \].

If C(S*)=2m, then \(|U|=2m+1\), and it is obvious that U is isomorphic to a cyclic group generated from the cyclic permutation (1 2 \( \ldots \) 2m+1). Thus, it follows from the Lemma 4.1 that the \( Ω(S) \) forms a group. Q.E.D.

For the number of states of automata, the necessary and sufficient condition to form the group can be derived, as follows.

**Theorem 4.2.** Let S=[Z, A]∈&^{[i≤2m+1]} (m≥1). Then, C(S*)=2m if and only if (1) \([2^m]_{2m+1}=1\) (2) for each \( i(1≤i≤2m-1) \) \([2^i]_{2m+1}≠1\).

Proof. Let S=[Z, A]∈&^{[i≤2m+1]} From proposition 4.1.2, cycle of S* is independent of permutation of their states. Therefore, without loss of generality, we can assume \( φ(A)=\{1 2 \ldots 2m+1\} \). Then, for each \( k≥1 \), we have

\[ φ(A)^{2k} = (1 f(k, 2) f(k, 3) \ldots f(k, 2m+1)) \]

where \( f(k, r)=[1+(r-1)2^k]_{2m+1} \) for each \( r(2≤r≤2m+1) \). It is obvious that for each \( k, j(i≥1) \), if \( f(k, 2)=f(j, 2) \), then \( φ(A)^{2k}=φ(A)^{2j} \). It follows from C(S*)=2m and Proposition 4.2.1 that

\[ φ(A)^{2m}=φ(A^{2m})=φ(A) \]

Thus, the condition can be derived as follows,

\[ [1+2^m]_{2m+1}=2, \]
\[ [1+2^1]_{2m+1}≠2, \text{for each } i=1, 2, \ldots, 2m-1. \]

Namely,

\[ [2^m]_{2m+1}=1, \text{and} \]
\[ [2^1]_{2m+1}≠1, \text{for each } i=1, 2, \ldots, 2m-1. \]

Converse is easily shown. Q.E.D.
As an example of Theorem 4.2, in Figure 2, we show one cycle of \( S^* \) of a CDAOA with five states.

![Diagrams of \( S^* \) and \( S^{16} \)](image)

**Fig. 2 One cycle of \( S^* \) (\( S=\{1,2,3,4,5\}, \ A\))**

Concerning a \( S^* \) of CDAOA \( S \) with \( 2^m \) (\( m \geq 1 \)) states, we obtain a theorem as follows.

**Theorem 4.3.** If \( S=\{Z, A\} \subseteq \&_{m}^{[n]} \) (\( m \geq 1 \)), then \( S^m = S^o(Z) \), and \( \lim_{n \to \infty} S^n = S^o(Z) \).

**Proof.** Let \( S=\{Z, A\} \subseteq \&_{m}^{[n]} \). From Proposition 2.2.2, there exists a SRF \( \tau \) such that \( \tau (A) \) is the canonical matrix, and

\[
\varphi(\tau(A)) = (1 \ 2 \ldots \ 2^m).
\]

The proof is derived by the induction on the number \( m \).

The initial step is trivial. \( ((1 \ 2) \cdot (1 \ 2) = (1) \odot (2)) \)

The induction step is taken from \( k \) to \( k+1 \). For \( \varphi(\tau(A)) = (1 \ 2 \ldots \ 2^k \cdot 1) \), we have

\[
\varphi(\tau(A))^{2^{k+1}} = (1 \ 3 \ 5 \ldots \ 2^{k+1}-1) \odot (2 \ 4 \ 6 \ldots \ 2^{k+1}) \quad \text{(by Theorem 3.3)}
\]

Let \( B=(1 \ 3 \ 5 \ldots \ 2^{k+1}-1), C=(2 \ 4 \ 6 \ldots \ 2^{k+1}) \). Nothing that the set \( \{1, 3, \ldots, 2^{k+1}-1\} \) and \( \{2, 4, 6, \ldots, 2^{k+1}\} \) are disjoint, we have

\[
\varphi(\tau(A))^{2^{k+1}} = \varphi(\tau(A))^{2^k} \odot B^{2^k} \odot C^{2^k}
\]

Since \( |\{1, 3, 5, \ldots, 2^{k+1}-1\}| = |\{2, 4, 6, \ldots, 2^{k+1}\}| = 2^k \), by the induction hypothesis,

\[
B^{2^k} = (1) \odot (3) \odot \cdots \odot (2^{k+1}-1)
\]

\[
C^{2^{k+1}} = (2) \odot (4) \odot \cdots \odot (2^{k+1}).
\]

Therefore,

\[
\varphi(\tau(A))^{2^{k+1}} = ((1) \odot (2) \odot (3) \odot \cdots \odot (2^{k+1})).
\]

From Proposition 2.3.2, we obtain \( A^{2^{k+1}} = I \), namely \( S^{2^{k+1}} = S^o(Z) \).

Then \( \lim_{n \to \infty} S^n = S^o(Z) \). Q. E. D.

Fig. 3 illustrates an example of Theorem 4.3.
5. Conclusion

As the automata model, whose aspects vary with the lapse of time, we introduce product automata and self-product automata. And some structural properties of product and self-product automata are analyzed by the product of state transition matrices.

In this paper, in order to simplify the problem, we deal with cyclic type of deterministic autonomous outputless automaton, however, it will be of interest to study the structural properties of general type of non-deterministic one.

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