

Dynamics of solitons in a damped sineHilbert equation

Y. Matsuno

Citation: *Journal of Mathematical Physics* **33**, 2754 (1992); doi: 10.1063/1.529544

View online: <http://dx.doi.org/10.1063/1.529544>

View Table of Contents: <http://scitation.aip.org/content/aip/journal/jmp/33/8?ver=pdfcov>

Published by the [AIP Publishing](#)



Re-register for Table of Content Alerts

Create a profile.



Sign up today!



Dynamics of solitons in a damped sine-Hilbert equation

Y. Matsuno

Department of Physics, Faculty of Liberal Arts, Yamaguchi University, Yamaguchi 753, Japan

(Received 29 October 1991; accepted for publication 18 March 1992)

A damped sine-Hilbert (sH) equation is proposed. It can be linearized by a dependent variable transformation which enables one to solve an initial value problem of the equation. The N -soliton solution is obtained explicitly and its properties are investigated in comparison with those of the N -soliton solution of the sH equation. In particular the interaction of the two solitons is explored in detail with the aid of the pole representation. It is found that the interaction process is classified into the two types according to the initial amplitudes and positions of both solitons. In the general N -soliton case the long-time behavior of the solution is shown to be characterized by the positive N zeros of the Hermite polynomial of degree $2N$. Finally, a linearized version of the damped sH equation is briefly discussed.

I. INTRODUCTION

In a series of papers¹⁻⁵ we have studied the analytical properties of the sine-Hilbert (sH) equation.^{6,7} A remarkable feature of the equation is that it can be linearized by an appropriate dependent variable transformation. This fact has enabled us to solve an initial value problem of the sH equation. As special cases, soliton^{1,2} and periodic³ solutions have been derived. It is also remarked that for a more general initial conditions, the sH equation has been solved by means of the inverse scattering method.⁸ A novel characteristic of one-soliton solution is that the propagation velocity is *inversely* proportional to the amplitude. Hence, a tall soliton propagates more slowly than a small one unlike the behavior of the usual soliton which propagates with the velocity proportional to its amplitude. Furthermore, in the interaction of N solitons only one soliton propagates with a constant velocity after multiple collisions of solitons, whereas the other $N-1$ solitons are decelerated with increasing amplitudes, and eventually in infinite time, the amplitudes blow up.² However, the occurrence of singularities in solutions would be unreasonable from the physical point of view. Therefore, it is quite natural to ask whether an addition of a damping term to the sH equation can suppress the blowup of solutions.

Motivated by the above-mentioned facts, we introduce the following damped sH equation

$$H\theta_t = -\sin \theta - \epsilon\theta_x, \quad \theta = \theta(x, t). \quad (1.1a)$$

Here the integral operator H defined by

$$H\theta(x, t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\theta(y, t)}{y-x} dy \quad (1.1b)$$

is the Hilbert transform, the subscripts t and x appended to θ denote partial differentiation and ϵ is a positive con-

stant representing the magnitude of the damping. The main reason why we have added a damping term of the form $-\epsilon\theta_x$ is that the equation is linearizable and consequently it can be solved *exactly* without any approximations. This remarkable aspect of the proposed equation should be stressed since to solve dissipative nonlinear evolution equations one must recourse to perturbation methods.^{9,10} However, the applicability of the methods is usually restricted to the system of equations with small perturbations. The famous Burgers equation in one-dimensional gas dynamics is an exception which can be linearized by means of the Hopf-Cole transformation. In this paper we solve Eq. (1.1) under the boundary condition $\theta_x \rightarrow 0$ as $|x| \rightarrow \infty$, which results in soliton solutions.

In Sec. II it is demonstrated that Eq. (1.1) can be linearized by introducing a dependent variable transformation. Using the pole representation of the solution, the linear partial differential equation thus obtained is reduced to a system of N ordinary differential equations. An explicit rational solution is then constructed which corresponds to the N -soliton solution of Eq. (1.1). In Sec. III the properties of the N -soliton solution are investigated for $N=1$, $N=2$, and general N separately. In particular the interaction of the two solitons is explored in detail with the aid of the pole representation. It is shown that the interaction process can be classified into the two types according to the initial amplitudes and positions of both solitons. The results are also compared with those for the sH equation [Eq. (1.1) with $\epsilon=0$]. For general N it is found that the asymptotic behavior of the N -soliton solution for large time is characterized by the positive N zeros of the Hermite polynomial of degree $2N$. In Sec. IV an initial value problem of a linearized version of Eq. (1.1) is solved by means of the Fourier transform and the characteristics of the solution are compared with those of the full nonlinear case treated in Sec. III. Section V is devoted to concluding remarks.

II. METHOD FOR EXACT SOLUTION

A. Linearization

Here we show that Eq. (1.1) can be linearized. Let us first introduce the following dependent variable transformation

$$\theta = i \ln(f_+/f_-), \tag{2.1}$$

where f_+ (f_-) is an analytic function with zeros in the lower (upper) complex plane. Substituting (2.1) into Eq. (1.1) and using the relation

$$H\theta_t = -[\ln(f_+f_-)]_x, \tag{2.2}$$

which stems from the analytical property of the functions f_{\pm} , Eq. (1.1) is transformed into the following bilinear equation for f_{\pm} :

$$(f_+f_-)_t = (1/2i)(f_-^2 - f_+^2) + i\epsilon(f_+x f_- - f_+f_-x). \tag{2.3}$$

Furthermore, Eq. (2.3) is modified in the form

$$f_- [f_{+,t} + (1/2i)(f_+ - f_-) - i\epsilon f_{+,x}] + f_+ [f_{-,t} - (1/2i)(f_- - f_+) + i\epsilon f_{-,x}] = 0. \tag{2.4}$$

Therefore, one sees that Eq. (2.4) is satisfied identically if the following system of equations for f_{\pm} holds

$$f_{+,t} + (1/2i)(f_+ - f_-) - i\epsilon f_{+,x} = i\lambda f_+, \tag{2.5a}$$

$$f_{-,t} - (1/2i)(f_- - f_+) + i\epsilon f_{-,x} = -i\lambda f_-, \tag{2.5b}$$

where $\lambda = \lambda(x, t)$ is a real function of x and t . In general λ will depend on f_{\pm} in a very complicated way except for a special case described in the following. The solution method of Eq. (2.5) for general λ is beyond the scope of the present paper.

Now, in order to obtain soliton solutions we set $\lambda = 0$ and assume the functional forms of f_{\pm} as

$$f_+ = f^*, \quad f_- = f, \tag{2.6a}$$

with

$$f = \prod_{j=1}^N [x - x_j(t)], \quad \text{Im } x_j > 0. \tag{2.6b}$$

Here x_j ($j=1, 2, \dots, N$) are complex functions of t whose imaginary parts are all positive and the asterisk denotes complex conjugate. It then turns out that Eqs. (2.5) are reduced to a single linear equation for f as follows:

$$f_t = (1/2i)(f - f^*) - i\epsilon f_x. \tag{2.7}$$

The initial value problem of Eq. (2.7) can be solved exactly provided that the initial profile $f(x, 0)$ is specified appropriately. In the present situation we first expand (2.6b) as

$$f = \sum_{j=0}^N (-1)^j s_j x^{N-j} \tag{2.8}$$

$$(s_0 = 1, \quad s_j = s_j(t), \quad j = 1, 2, \dots, N),$$

where s_j are elementary symmetric functions of x_1, x_2, \dots, x_N :

$$s_1 = \sum_{j=1}^N x_j, \tag{2.9a}$$

$$s_2 = \sum_{j < k}^N x_j x_k, \tag{2.9b}$$

$$\vdots$$

$$s_N = \prod_{j=1}^N x_j. \tag{2.9c}$$

Then we substitute (2.8) into (2.7) and compare the coefficients of x^{N-j} on both sides. Thus we arrive at the following system of linear ordinary differential equations for s_j :

$$\dot{s}_j = \text{Im } s_j + i\epsilon(N - j + 1)s_{j-1} \quad (j = 1, 2, \dots, N). \tag{2.10}$$

Here an overdot appended to s_j denotes the time differentiation. The above system of equations is a main result in this paper. We now specify the initial values of x_j and s_j in the forms

$$x_j(0) = \tilde{b}_j + i\tilde{a}_j \quad (j = 1, 2, \dots, N), \tag{2.11a}$$

$$s_j(0) = b_j + ia_j \quad (j = 1, 2, \dots, N), \tag{2.11b}$$

where $\tilde{a}_j, \tilde{b}_j, a_j$ and b_j are real constants. It should be noted that the conditions $\tilde{a}_j > 0$ ($j=1, 2, \dots, N$) must be satisfied to fulfill the analytical requirement for x_j [see (2.6b)]. One can see from (2.9) that these constants are related to each other as

$$a_1 = \sum_{j=1}^N \tilde{a}_j, \quad b_1 = \sum_{j=1}^N \tilde{b}_j \quad (a_1 > 0), \tag{2.12a}$$

$$a_2 = \sum_{j < k}^N (\tilde{a}_j \tilde{b}_k + \tilde{a}_k \tilde{b}_j) \quad b_2 = \sum_{j < k}^N (-\tilde{a}_j \tilde{a}_k + \tilde{b}_j \tilde{b}_k), \tag{2.12b}$$

$$\vdots$$

$$a_N = \text{Im} \left[\prod_{j=1}^N (\tilde{b}_j + i\tilde{a}_j) \right], \quad b_N = \text{Re} \left[\prod_{j=1}^N (\tilde{b}_j + i\tilde{a}_j) \right]. \tag{2.12c}$$

Next we derive the equation of motion for x_j ($j = 1, 2, \dots, N$). A simple way is to divide (2.7) by f and then compare the coefficient of $(x - x_j)^{-1}$ on both sides. The resulting expressions read in the forms

$$\dot{x}_j = \frac{1}{2i} \left[\prod_{k=1}^N (x_j - x_k^*) \left(\prod_{\substack{k=1 \\ (k \neq j)}}^N (x_j - x_k) \right)^{-1} \right] + i\epsilon \quad (j=1, 2, \dots, N). \tag{2.13}$$

Hence, the problem under consideration has been reduced to solve dynamical motions of N complex variables x_1, x_2, \dots, x_N . However, in the present case the solutions for Eqs. (2.13) are obtained directly by solving the algebraic equation of degree N , $f=0$ where f is given by (2.8). If we take the imaginary part of Eqs. (2.13), we have

$$\text{Im } \dot{x}_j = G_j \text{Im } x_j + \epsilon \quad (j=1, 2, \dots, N), \tag{2.14a}$$

with real functions defined by

$$G_j = \text{Im} \prod_{\substack{k=1 \\ (k \neq j)}}^N \frac{x_j - x_k^*}{x_j - x_k}. \tag{2.14b}$$

Integrating (2.14) yields the important relations

$$\begin{aligned} \text{Im } x_j(t) &= \left[\text{Im } x_j(0) + \epsilon \int_0^t dt' \exp\left(-\int_0^{t'} G_j(t'') dt''\right) \right] \\ &\quad \times \exp\left(\int_0^t G_j(t') dt'\right) \quad (j=1, 2, \dots, N). \end{aligned} \tag{2.15}$$

Equations (2.15) would ensure that if the conditions $\text{Im } x_j(0) > 0$ ($j=1, 2, \dots, N$) are satisfied, then the same inequalities hold for a later time, i.e., $\text{Im } x_j(t) > 0$ ($t > 0, j=1, 2, \dots, N$). Thus the analytical condition for f necessary in deriving (2.2) would be fulfilled for all positive time.

The solution (2.1) with (2.6) may be called the N -kink solution by analogy with kink solutions of the sine-Gordon equation; this is easily seen by rewriting (2.1) in the form $\theta = i \ln(f^*/f) = 2 \tan^{-1}(\text{Im } f / \text{Re } f)$. However, it is more convenient to introduce the function $u \equiv \theta_x$ instead of θ to visualize the solution

$$u = \sum_{j=1}^N \left(\frac{i}{x - x_j^*} - \frac{i}{x - x_j} \right) = \sum_{j=1}^N u_j, \tag{2.16a}$$

where

$$u_j = \frac{2 \text{Im } x_j}{(x - \text{Re } x_j)^2 + (\text{Im } x_j)^2} \quad (j=1, 2, \dots, N). \tag{2.16b}$$

The expression (2.16) is the so-called pole representation of the N -soliton solution. In what follows we term u_j the j th soliton. In the pole representation the j th soliton may be regarded as a particle located in the complex plane. In the terminology of particle physics the interaction of solitons is interpreted as inelastic scatterings between particles since in the present situation, the dissipation of energy is accompanied due to the effect of the damping.

B. Exact solutions

In this section we solve Eq. (2.10). It follows from the real and imaginary parts of Eq. (2.10) that

$$\text{Re } \dot{s}_j = \text{Im } s_j - \epsilon(N - j + 1) \text{Im } s_{j-1}, \tag{2.17a}$$

$$\text{Im } \dot{s}_j = \epsilon(N - j + 1) \text{Re } s_{j-1}. \tag{2.17b}$$

We now expand $\text{Re } s_j$ and $\text{Im } s_j$ in powers of t as

$$\text{Re } s_j = \sum_{s=0}^{2j} c_s^{(j)} t^{2j-s}, \tag{2.18a}$$

$$\text{Im } s_j = \sum_{s=0}^{2j-1} d_s^{(j)} t^{2j-s-1}, \tag{2.18b}$$

where $c_s^{(j)}$ and $d_s^{(j)}$ are constants to be determined. Substituting (2.18) into (2.17) and comparing the coefficients of the same powers of t on both sides, we obtain the recursion relations for $c_s^{(j)}$ and $d_s^{(j)}$ as follows:

$$\begin{aligned} (2j-s)c_s^{(j)} &= d_s^{(j)} - \epsilon(N-j+1)d_{s-2}^{(j-1)} \\ &\quad (s=2, 3, \dots, 2j-1; \quad j=2, 3, \dots, N), \end{aligned} \tag{2.19a}$$

$$(2j-1)c_1^{(j)} = d_1^{(j)} \quad (j=1, 2, \dots, N), \tag{2.19b}$$

$$2jc_0^{(j)} = d_0^{(j)} \quad (j=1, 2, \dots, N), \tag{2.19c}$$

$$\begin{aligned} (2j-s-1)d_s^{(j)} &= \epsilon(N-j+1)c_s^{(j-1)} \\ &\quad (s=0, 1, \dots, 2j-2; \quad j=1, 2, \dots, N). \end{aligned} \tag{2.19d}$$

These relations must be solved under the conditions

$$c_{2j}^{(j)} = b_j, \quad d_{2j-1}^{(j)} = a_j \quad (j=1, 2, \dots, N), \tag{2.20}$$

which are derived from (2.11b) and (2.18). In the absence of the damping ($\epsilon=0$), the solution for Eqs. (2.19) is easily found and it is simply expressed as $s_j = a_j t + b_j + ia_j$ ($j=1, 2, \dots, N$). This corresponds to the rational N -

soliton solution of the sH equation.^{1,2} Let us now solve Eqs. (2.19) for even and odd s separately.

1. $s=2m$; $m=1,2,\dots,j-1$

In this case (2.19a) and (2.19d) yield

$$(2j-2m)c_{2m}^{(j)} = \frac{\epsilon(N-j+1)}{2j-2m-1} c_{2m}^{(j-1)} - \epsilon^2 \frac{(N-j+1)(N-j+2)}{2j-2m-1} c_{2m-2}^{(j-2)}. \tag{2.21}$$

If we put

$$c_{2m}^{(j)} = \frac{(N-m)! \epsilon^{j-m}}{(2j-2m)!(N-j)!} \tilde{c}_{2m}^{(j)} \quad (\tilde{c}_{2j}^{(j)} = b_j), \tag{2.22}$$

Eq. (2.21) is considerably simplified as

$$\tilde{c}_{2m}^{(j)} = \tilde{c}_{2m}^{(j-1)} - \epsilon(N-m+1) \tilde{c}_{2m-2}^{(j-2)}. \tag{2.23}$$

The solution of the above recursion relation is expressed in the form

$$\tilde{c}_{2m}^{(j)} = b_m + \sum_{k=1}^{j-m} \left[(-\epsilon)^k \binom{j-m}{k} \prod_{l=1}^k (N-m+l) \right] b_{m-k} \tag{2.24}$$

where $\binom{j-m}{k}$ is a binomial coefficient. In (2.24) we have assumed $b_0=1$ and $b_j=0$ ($j < 0$). The solution (2.24) can be proved by a mathematical induction using the identity $\binom{j+1}{m} = \binom{j}{m} + \binom{j}{m-1}$ of the binomial coefficients. If we employ (2.19d), the coefficient $d_{2m}^{(j)}$ is written in terms of $\tilde{c}_{2m}^{(j-1)}$ as

$$d_{2m}^{(j)} = \frac{\epsilon(N-j+1)}{2j-2m-1} c_{2m}^{(j-1)} = \frac{(N-m)! \epsilon^{j-m}}{(2j-2m-1)!(N-j)!} \tilde{c}_{2m}^{(j-1)}. \tag{2.25}$$

Equation (2.19c) is then automatically satisfied by (2.22) and (2.25).

2. $s=2m+1$; $m=1,2,\dots,j-1$

In this case it follows from (2.19a) and (2.19d) that

$$(2j-2m-1)d_{2m+1}^{(j+1)} = \frac{\epsilon(N-j)}{2j-2m} d_{2m+1}^{(j)} - \epsilon^2 \frac{(N-j)(N-j+1)}{2j-2m} d_{2m-1}^{(j-1)}. \tag{2.26}$$

Introduction of a new variable $\tilde{d}_{2m-1}^{(j)}$ by

$$d_{2m-1}^{(j)} = \frac{(N-m)! \epsilon^{j-m}}{(2j-2m)!(N-j)!} \tilde{d}_{2m-1}^{(j)} \quad (\tilde{d}_{2j-1}^{(j)} = a_j), \tag{2.27}$$

transforms (2.26) into the following recursion relation

$$\tilde{d}_{2m+1}^{(j+1)} = \tilde{d}_{2m+1}^{(j)} - \epsilon(N-m) \tilde{d}_{2m-1}^{(j-1)}. \tag{2.28}$$

One finds the solution in the form

$$\tilde{d}_{2m-1}^{(j)} = a_m + \sum_{k=1}^{j-m} \left[(-\epsilon)^k \binom{j-m}{k} \prod_{l=1}^k (N-m+l) \right] a_{m-k} \tag{2.29}$$

where we have assumed $a_j=0$ ($j \leq 0$). Then $c_{2m-1}^{(j)}$ is given by

$$c_{2m-1}^{(j)} = \frac{2j-2m+2}{\epsilon(N-j)} d_{2m-1}^{(j+1)} = \frac{(N-m)! \epsilon^{j-m}}{(2j-2m+1)!(N-j)!} \tilde{d}_{2m-1}^{(j+1)}. \tag{2.30}$$

It is easy to confirm that Eq. (2.19b) is satisfied by (2.27) and (2.30). For later use we write down the explicit forms of s_j for $N=1, 2$:

$$N=1: s_1 = (\epsilon/2)t^2 + a_1 t + b_1 + i(\epsilon t + a_1); \tag{2.31}$$

$$N=2: s_1 = \epsilon t^2 + a_1 t + b_1 + i(2\epsilon t + a_1), \tag{2.32a}$$

$$s_2 = \epsilon \left[\frac{\epsilon}{12} t^4 + \frac{a_1}{6} t^3 + \frac{1}{2} (b_1 - 2\epsilon) t^2 \right] + (a_2 - \epsilon a_1) t + b_2 + i \left[\epsilon \left(\frac{\epsilon}{3} t^3 + \frac{a_1}{2} t^2 + b_1 t \right) + a_2 \right]. \tag{2.32b}$$

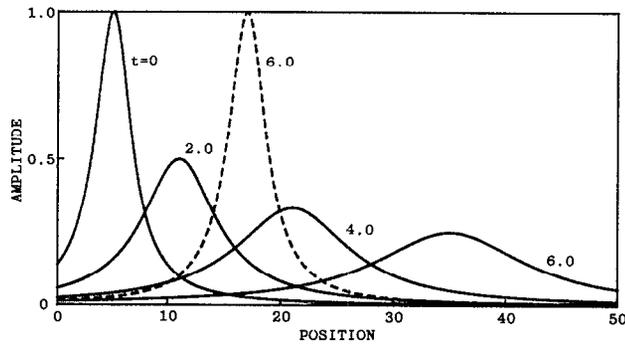


FIG. 1. The time evolution of one-soliton solution. The broken line shows the corresponding profile at $t=6.0$ in the absence of the damping.

III. PROPERTIES OF SOLUTIONS

A. $N=1$

It follows from (2.9a), (2.16), and (2.31) that

$$u = \frac{2 \operatorname{Im} x_1}{(x - \operatorname{Re} x_1)^2 + (\operatorname{Im} x_1)^2}, \tag{3.1a}$$

with

$$\operatorname{Re} x_1 = (\epsilon/2)t^2 + a_1 t + b_1 \quad (a_1 > 0), \tag{3.1b}$$

$$\operatorname{Im} x_1 = \epsilon t + a_1. \tag{3.1c}$$

The one-soliton solution represents a pulse moving to the right direction with an amplitude $2/(\epsilon t + a_1)$ and a velocity $\epsilon t + a_1$. This implies that the propagation velocity is inversely proportional to its amplitude. However, because of the effect of the damping, the amplitude decreases with time so that the velocity increases indefinitely as time goes. Figure 1 represents the typical profile of u at $t=0, 2.0, 4.0, 6.0$. The parameters are chosen as $a_1=2.0, b_1=5.0, \epsilon=1.0$. The broken line in the figure also shows the profile of u at $t=6.0$ in the absence of the damping, i.e., $\epsilon=0$.

In Fig. 2 the trajectories of x_1 are plotted in the complex plane for various values of ϵ where the values of a_1 and b_1 are the same as those used in Fig. 1. The arrow in the figure indicates the direction of the motion and a small circle shows the initial position of the pole, $x_1 = (5.0, 2.0)$ in the present example. The straight line corresponds to the case of $\epsilon=0$. For $\epsilon \neq 0$ each trajectory is deformed by the damping and it approaches the parabola $\operatorname{Re} x_1 = (\operatorname{Im} x_1)^2 / 2\epsilon$ after a long lapse of time.

B. $N=2$

In this case we find that the two-soliton solution is represented by a superposition of two one-soliton solutions as

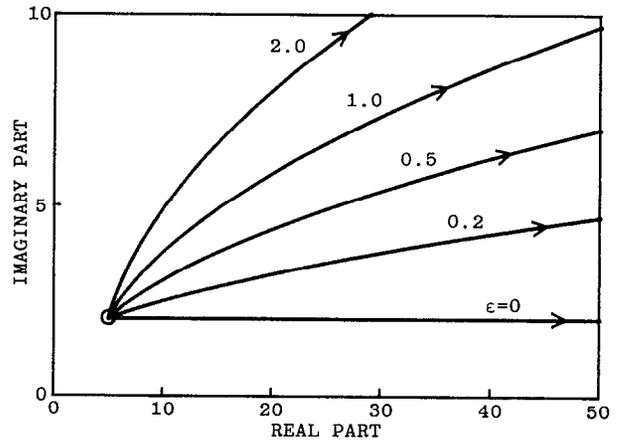


FIG. 2. The trajectories of the pole for various values of ϵ . The arrow indicates the direction of the motion and a small circle shows the initial position of the pole.

$$u = \sum_{j=1}^2 \frac{2 \operatorname{Im} x_j}{(x - \operatorname{Re} x_j)^2 + (\operatorname{Im} x_j)^2}. \tag{3.2}$$

Here x_j ($j=1,2$) are obtained by solving the algebraic equation of degree two:

$$x^2 - s_1 x + s_2 = 0, \tag{3.3}$$

where s_1 and s_2 are already given by (2.32).

First of all we define the following quantities to simplify the notation:

$$A = \epsilon t^2 + a_1 t + b_1, \tag{3.4a}$$

$$B = 2\epsilon t + a_1, \tag{3.4b}$$

$$C = \frac{2}{3}\epsilon^2 t^4 + \frac{4}{3}\epsilon a_1 t^3 + a_1^2 t^2 + (2a_1 b_1 - 4a_2)t - a_1^2 - 4b_2 + b_1^2, \tag{3.4c}$$

$$D = 2(\frac{4}{3}\epsilon^2 t^3 + 2\epsilon a_1 t^2 + a_1^2 t + a_1 b_1 - 2a_2). \tag{3.4d}$$

The interaction process of two solitons may be classified by the initial amplitudes and positions of both solitons. The following two cases arise, which we consider separately.

1. Case 1

Case 1 is characterized by the conditions

$$a_1 b_1 - 2a_2 = (\tilde{a}_1 - \tilde{a}_2)(\tilde{b}_1 - \tilde{b}_2) < 0, \tag{3.5a}$$

$$C(t_0) > 0, \tag{3.5b}$$

where t_0 is a positive root of the algebraic equation $D=0$. Its existence is obvious from (3.4d) and (3.5a). In (3.5a)

\tilde{b}_j and $2/\tilde{a}_j$ represent the initial position and amplitude of the j th soliton. In what follows we assume $\tilde{a}_1 > \tilde{a}_2$ without loss of generality.

Now x_1 and x_2 are readily obtained from (3.3) as follows:

$$x_1 = \frac{1}{2} \left[A - \sqrt{\frac{1}{2} (C + \sqrt{C^2 + D^2})} + i \left(B - \frac{D}{|D|} \sqrt{\frac{1}{2} (-C + \sqrt{C^2 + D^2})} \right) \right], \quad (3.6a)$$

$$x_2 = \frac{1}{2} \left[A + \sqrt{\frac{1}{2} (C + \sqrt{C^2 + D^2})} + i \left(B + \frac{D}{|D|} \sqrt{\frac{1}{2} (-C + \sqrt{C^2 + D^2})} \right) \right]. \quad (3.6b)$$

Upon expanding the quantities $A-D$ in powers of t , we find the asymptotic behaviors of x_1 and x_2 for large time in the forms

$$x_1 \sim \frac{1}{2} (1 - \sqrt{2/3}) \epsilon t^2 + i (1 - \sqrt{2/3}) \epsilon t, \quad (3.7a)$$

$$x_2 \sim \frac{1}{2} (1 + \sqrt{2/3}) \epsilon t^2 + i (1 + \sqrt{2/3}) \epsilon t. \quad (3.7b)$$

An important observation is that in the leading order of the expansion these expressions do not depend on the parameters a_j and b_j ($j=1, 2$). In Sec. III C we show that this property also survives for general N -soliton solution. On the other hand, in the absence of the damping the corresponding expressions are written in the forms

$$x_1 \sim a_2/a_1 + i a_1^{-3} (a_1 a_2 b_1 - a_1^2 b_2 - a_2^2) t^{-2}, \quad (3.8a)$$

$$x_2 \sim a_1 t + i a_1. \quad (3.8b)$$

In the pole representation of the solution the distance $l=l(t)$ between the two poles is given by

$$l \equiv |x_1(t) - x_2(t)| = (C^2 + D^2)^{1/4}. \quad (3.9)$$

It is seen that l takes one minimum value at which instant the two solitons collide.

The typical profiles of u_1 , u_2 , and $u (=u_1 + u_2)$ are depicted in Fig. 3(a)–(c) at $t=0, 0.447, 1.0$. The values of the parameters are $a_1=12.5, a_2=-87.5, b_1=-20.0, b_2=50.0, \epsilon=1.0$ or equivalently $\tilde{a}_1=10.0, \tilde{a}_2=2.5, \tilde{b}_1=-15.0, \tilde{b}_2=-5.0$. One then sees that $a_1 b_1 - 2a_2 = -75.0, t_0=0.447$, and $C(t_0)=9.44$, so that the conditions (3.5) are satisfied. The corresponding profile of u takes the form

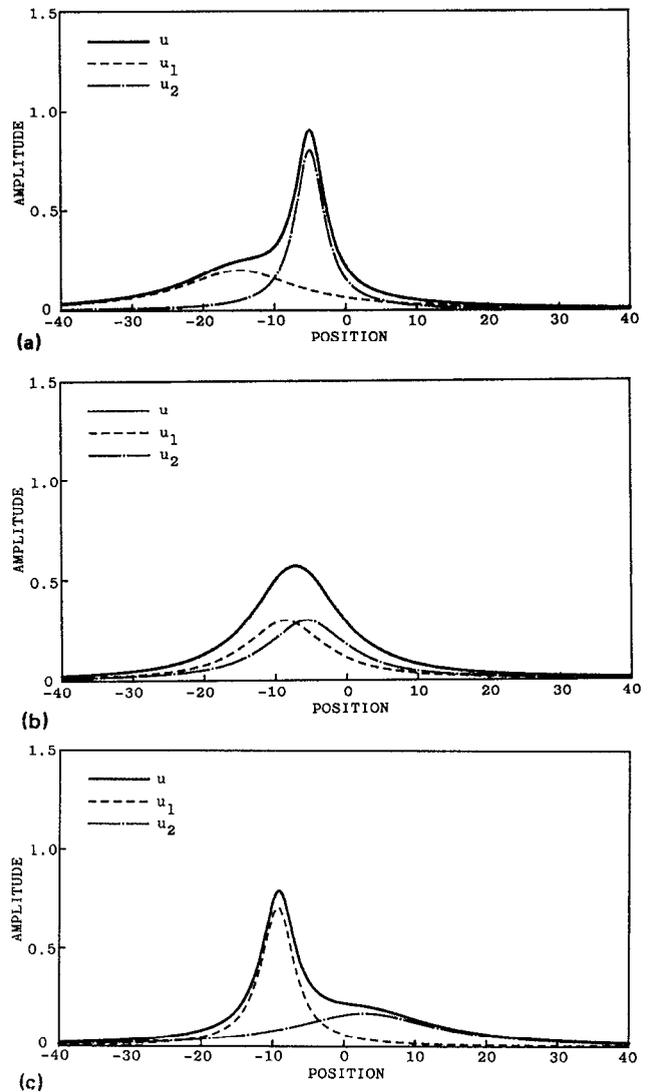


FIG. 3. The time evolution of two-soliton solution u with that of u_1 and u_2 in Case 1. (a) $t=0$, (b) $t=0.447$, (c) $t=1.0$.

$$u(x,0) = u_1(x,0) + u_2(x,0)$$

$$= \frac{20.0}{(x+15.0)^2 + 100.0} + \frac{5.0}{(x+5.0)^2 + 6.25}. \quad (3.10)$$

The trajectories of x_1 and x_2 in the complex plane are also drawn in Fig. 4 with the same parameter values. The initial positions of poles indicated by small circles are $x_1(0) = (-15.0, 10.0)$ and $x_2(0) = (-5.0, 2.5)$ in the present example. Note that the asymptotic behaviors of the two poles are given by (3.7).

The interaction process of the two solitons is now summarized as follows: As the two solitons get close, the smaller soliton u_1 increases in height and becomes thinner

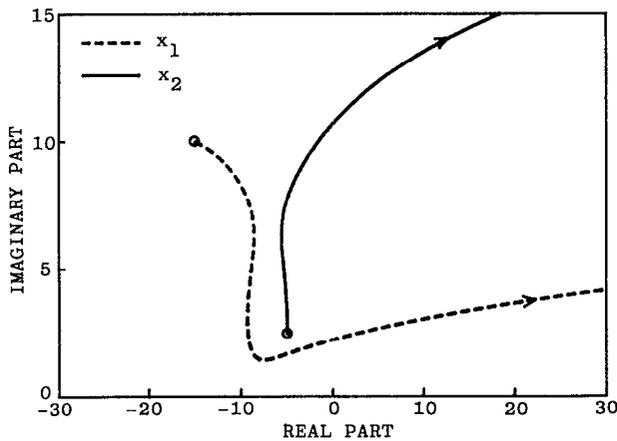


FIG. 4. The trajectories of the two poles in Case 1.

while the taller soliton u_2 decreases in height and grows flatter. At the instant of the collision, $t=0.447$ in the present example, the two solitons never coalesce and have the same profile as seen from Fig. 3(b). After the collision of the two solitons, the velocities of both solitons have been interchanged so that u_2 propagates faster than u_1 does [see Fig. 3(c)]. As time goes on, the two solitons are separated more and more with increasing velocities and asymptotically for a long time they behave like independent solitons. In order to compare the results with those for the case $\epsilon=0$, the figures corresponding to Fig. 3(a)–(c) and Fig. 4 are depicted in Fig. 5(a)–(c) and Fig. 6, respectively. In these figures the initial conditions are the same as those for the case $\epsilon=1.0$. In the case $\epsilon=0$, the collision of the two solitons occurs at $t=0.480$ [see Fig. 5(a)]. It is worthwhile to note that the asymptotic behavior of u for large time is given by

$$u \sim 2\pi\delta\left(x - \frac{a_2}{a_1}\right) + \frac{2a_1}{(x - a_1t)^2 + a_1^2}, \quad (3.11)$$

where δ is Dirac's delta function.² Therefore, u_1 blows up at the position $x = a_2/a_1 = -7.0$ while u_2 propagates with an amplitude $2/a_1 = 0.16$ and a velocity $a_1 = 12.5$ [see Fig. 5(c)].

2. Case 2

Case 2 is characterized by either the conditions

$$a_1b_1 - 2a_2 < 0, \quad (3.12a)$$

$$C(t_0) < 0, \quad (3.12b)$$

or the condition

$$a_1b_1 - 2a_2 \geq 0. \quad (3.13)$$

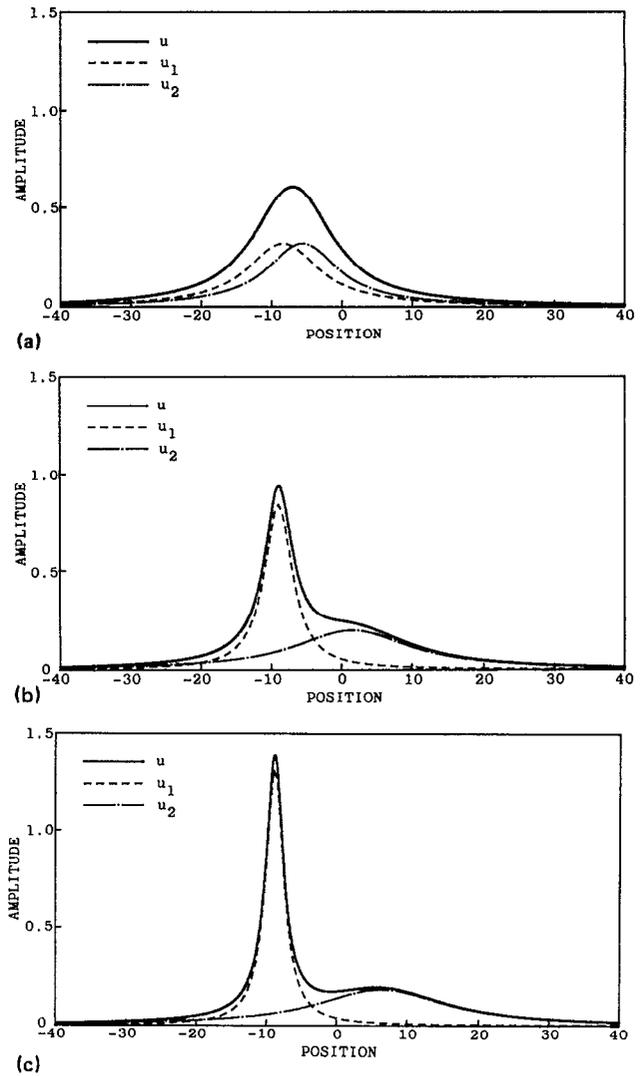


FIG. 5. The profiles of u , u_1 , and u_2 which correspond to Fig. 3 in the absence of the damping. (a) $t=0.480$, (b) $t=1.0$, (c) $t=1.4$.

In (3.13) there are no limitations on the sign of $C(t_0)$. The x_1 and x_2 are then given by the expressions

$$x_1 = \frac{1}{2} \left[A + \frac{D}{|D|} \sqrt{\frac{1}{2}(C + \sqrt{C^2 + D^2})} + i \left(B + \sqrt{\frac{1}{2}(-C + \sqrt{C^2 + D^2})} \right) \right], \quad (3.14a)$$

$$x_2 = \frac{1}{2} \left[A - \frac{D}{|D|} \sqrt{\frac{1}{2}(C + \sqrt{C^2 + D^2})} + i \left(B - \sqrt{\frac{1}{2}(-C + \sqrt{C^2 + D^2})} \right) \right], \quad (3.14b)$$

together with their long time behaviors

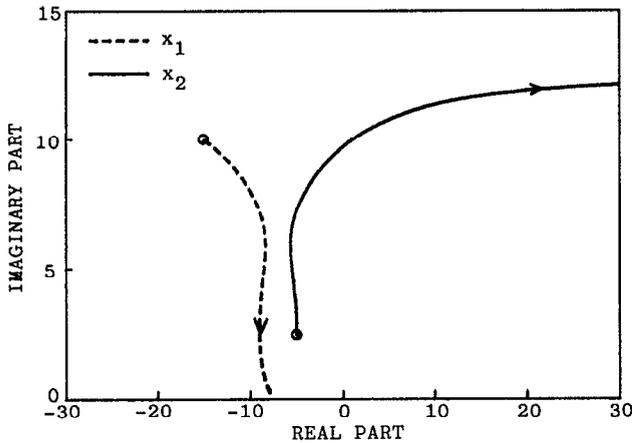


FIG. 6. The trajectories of the two poles which correspond to Fig. 4 in the absence of the damping.

$$x_1 \sim \frac{1}{2}(1 + \sqrt{2/3})\epsilon t^2 + i(1 + \sqrt{2/3})\epsilon t, \quad (3.15a)$$

$$x_2 \sim \frac{1}{2}(1 - \sqrt{2/3})\epsilon t^2 + i(1 - \sqrt{2/3})\epsilon t. \quad (3.15b)$$

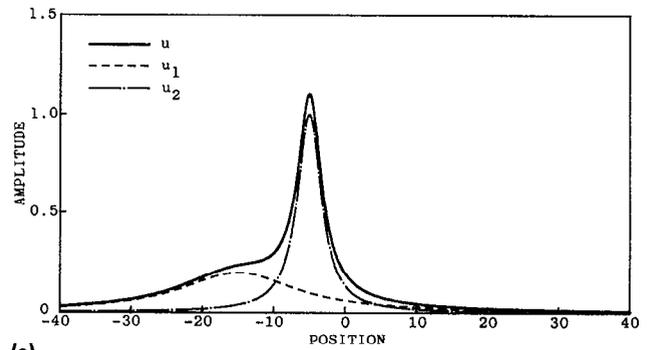
In the case $\epsilon=0$ the corresponding asymptotic expressions read in the forms

$$x_1 \sim a_1 t + i a_1, \quad (3.16a)$$

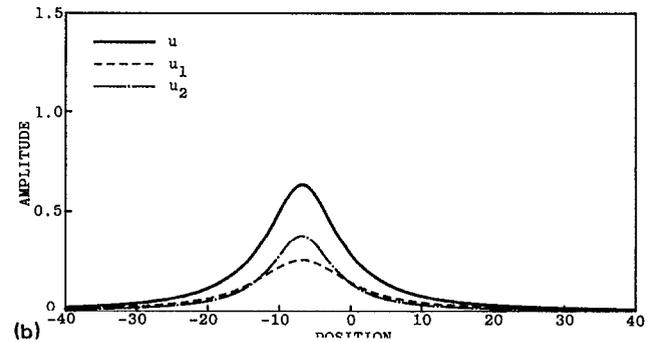
$$x_2 \sim a_2/a_1 + i a_1^{-3}(a_1 a_2 b_1 - a_1^2 b_2 - a_2^2) t^{-2}. \quad (3.16b)$$

The typical profiles of u_1 , u_2 , and u are shown in Fig. 7(a)–(c) at $t=0, 0.511, 1.0$ and those of x_1 and x_2 are plotted in Fig. 8. The parameters are chosen as $a_1=12.0$, $a_2=-80.0$, $b_1=-20.0$, $b_2=55.0$, $\epsilon=1.0$ ($\tilde{a}_1=10.0$, $\tilde{a}_2=2.0$, $\tilde{b}_1=-15.0$, $\tilde{b}_2=-5.0$). Then $a_1 b_1 - 2a_2 = -80.0$, $t_0=0.511$ and $C(t_0) = -5.98$ which satisfy the conditions (3.12). We do not consider here the case corresponding to the condition (3.13) since under the assumption $\tilde{a}_1 > \tilde{a}_2$, \tilde{b}_1 must be greater than \tilde{b}_2 so that the faster soliton always locates to the right of the slower soliton at the initial time. As a result the collision of the two solitons never occurs.

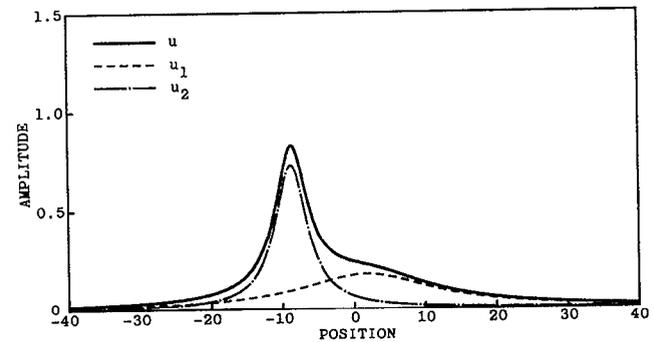
Now the interaction process of the two solitons is summarized as follows: As the two solitons get close, the smaller soliton absorbs the taller one. At the instant of the collision, $t=0.511$ in the present example, they coalesce into a single pulse [see Fig. 7(b)]. After a lapse of time, a taller soliton is emitted backward and eventually for a long time, the two independent solitons are formed. Therefore, in this case the two solitons pass through to each other in a collision process and the situation is in a striking contrast to that of Case 1. The corresponding figures in the absence of the damping are also presented in Fig. 9(a)–(c) and Fig. 10. In this case the collision occurs at $t=0.556$. The asymptotic expression of u for large time takes the same form as (3.11).



(a)



(b)



(c)

FIG. 7. The time evolution of two-soliton solution u with that of u_1 and u_2 in Case 2. (a) $t=0$, (b) $t=0.511$, (c) $t=1.0$.

C. General N

In this section we investigate the properties of general N -soliton solution. In particular we focus our attention on the asymptotic behaviors of the solution for large time and show that the amplitude and the position of each soliton are expressed in terms of the zeros of the Hermite polynomial. It now follows from (2.18), (2.22), (2.24), and (2.25) that in the leading order of the large time expansion, s_j are approximated by

$$s_j \sim c_0^{(j)} t^{2j} + i d_0^{(j)} t^{2j-1} \quad (j=1, 2, \dots, N), \quad (3.17a)$$

with

$$c_0^{(j)} = \frac{N! e^j}{(2j)!(N-j)!}, \quad (3.17b)$$

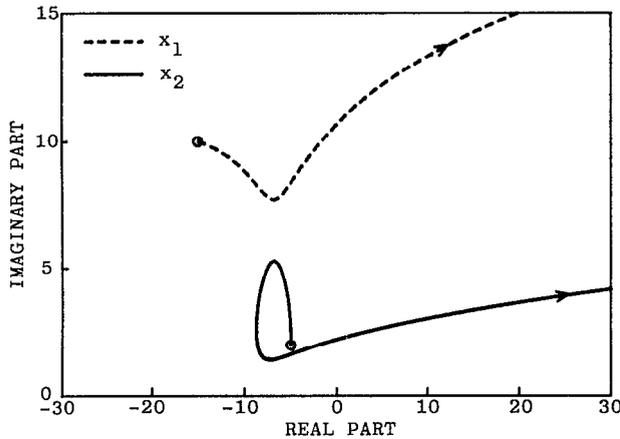


FIG. 8. The trajectories of the two poles in Case 2.

$$d_0^{(j)} = \frac{N! \epsilon^j}{(2j-1)!(N-j)!} \tag{3.17c}$$

The corresponding expression of u reads in the form

$$f \sim x^N + \sum_{j=1}^N \frac{(-1)^j N!}{(2j)!(N-j)!} \epsilon^{j2} x^{N-j} + i \sum_{j=1}^N \frac{(-1)^j N!}{(2j-1)!(N-j)!} \epsilon^{j2j-1} x^{N-j} \tag{3.18}$$

Suggested by the asymptotic expression of x_1 and x_2 [see (3.7) and (3.15)], we seek the solution of the algebraic equation $f=0$ in the form

$$x \sim \epsilon(\alpha t^2 + i\beta t) \tag{3.19}$$

Substituting (3.19) into (3.18) and taking the coefficients of t^{2N} and t^{2N-1} zero, we obtain the following two equations which determine α and β :

$$H(\alpha) \equiv \alpha^N + \sum_{j=1}^N \frac{(-1)^j N!}{(2j)!(N-j)!} \alpha^{N-j} = 0, \tag{3.20}$$

$$\left(N\alpha^{N-1} + \sum_{j=1}^{N-1} \frac{(-1)^j N!}{(2j)!(N-j-1)!} \alpha^{N-j-1} \right) \beta + \sum_{j=1}^N \frac{(-1)^j N!}{(2j-1)!(N-j)!} \alpha^{N-j} = 0. \tag{3.21}$$

It then turns out from (3.21) that

$$\beta = -L(\alpha)/H'(\alpha), \tag{3.22a}$$

where

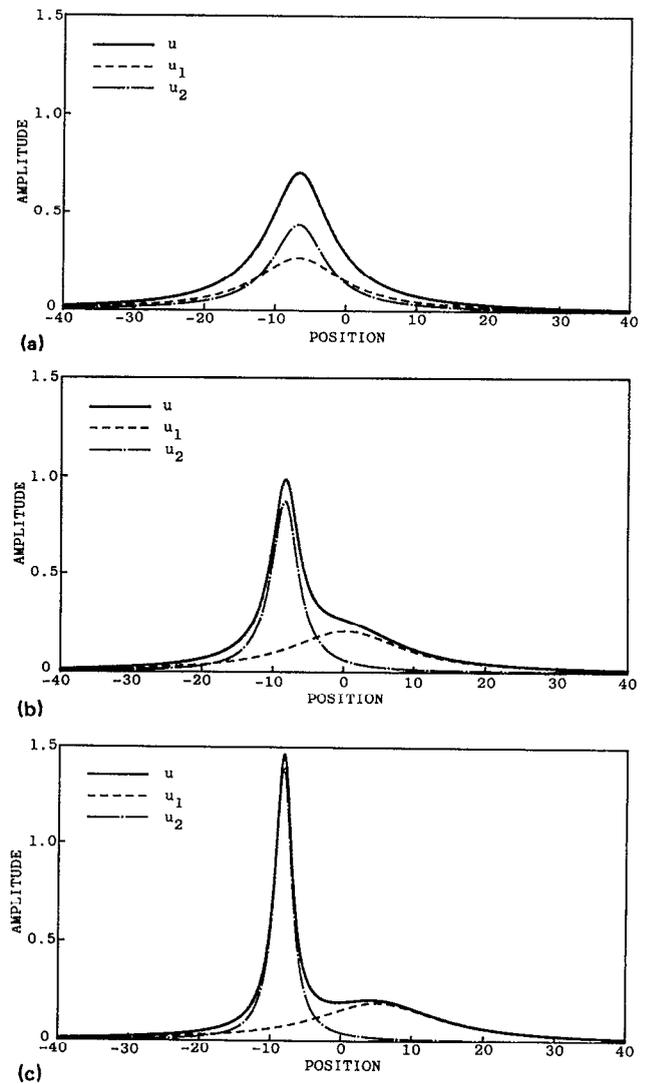


FIG. 9. The profiles of u , u_1 , and u_2 which correspond to Fig. 7 in the absence of the damping. (a) $t=0.556$, (b) $t=1.0$, (c) $t=1.4$.

$$L(\alpha) = \sum_{j=1}^N \frac{(-1)^j N!}{(2j-1)!(N-j)!} \alpha^{N-j}, \tag{3.22b}$$

and the prime appended to H denotes the differentiation with respect to α . Hence, the long time behaviors of x_j are characterized in the leading order of the expansion by the zeros of the algebraic equation (3.20).

At this stage we remember the definition of the Hermite polynomial

$$H_n(x) = \sum_{j=0}^{[n/2]} \frac{(-1)^j n!}{j!(n-2j)!} \frac{x^{n-2j}}{2^j}, \tag{3.23}$$

with its properties

$$H'_n(x) = nH_{n-1}(x), \tag{3.24a}$$

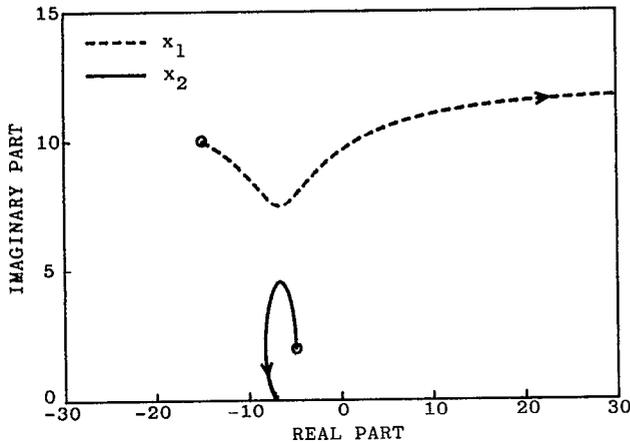


FIG. 10. The trajectories of the two poles which correspond to Fig. 8 in the absence of the damping.

$$H_{n+1}(x) - xH_n(x) = -nH_{n-1}(x), \tag{3.24b}$$

where $[n/2]$ implies the integer part of $n/2$. If we compare (3.20) with (3.23), we notice the relation

$$H(\alpha) = \frac{(-1)^{N!}}{(2N)!} (2\alpha)^N H_{2N}(1/\sqrt{2\alpha}). \tag{3.25}$$

Therefore, the N roots of Eq. (3.20) coincide with the positive N roots of the algebraic equation of degree $2N$

$$H_{2N}(1/\sqrt{2\alpha}) = 0. \tag{3.26}$$

On the other hand, the function $L(\alpha)$ defined in (3.22b) is represented by

$$L(\alpha) = \frac{(-1)^{N!}}{(2N-1)!} (2\alpha)^{N-1/2} H_{2N-1}(1/\sqrt{2\alpha}). \tag{3.27}$$

It is now possible to rewrite β in a more transparent form. To do so we differentiate (3.25) by α and then use (3.24a,b) to derive

$$H'(\alpha) = \frac{(-1)^{N+1}N!}{(2N-2)!} (2\alpha)^{N-1} H_{2N-2}(1/\sqrt{2\alpha}). \tag{3.28}$$

Substitution of (3.27) and (3.28) into (3.22) yields

$$\beta = \frac{\sqrt{2\alpha}}{2N-1} \frac{H_{2N-1}(1/\sqrt{2\alpha})}{H_{2N-2}(1/\sqrt{2\alpha})}. \tag{3.29}$$

Furthermore, if we put $n=2N-1$ and $x = 1/\sqrt{2\alpha}$ in (3.24b) and use (3.26), we obtain

$$\frac{H_{2N-1}(1/\sqrt{2\alpha})}{H_{2N-2}(1/\sqrt{2\alpha})} = (2N-1)\sqrt{2\alpha}. \tag{3.30}$$

Finally, it follows from (3.29) and (3.30) that

$$\beta = 2\alpha, \tag{3.31}$$

which is the desired relation. Let the positive N zeros of $H_{2N}(x)$ be $x_{j,N}$ ($j=1,2,\dots,N$). Then from the above argument the asymptotic forms of x_j are found to be

$$x_j \sim \epsilon(\alpha_j t^2 + i\beta_j t) \quad (j=1,2,\dots,N), \tag{3.32a}$$

with

$$\alpha_j = \frac{1}{2(x_{j,N})^2}, \quad \beta_j = \frac{1}{(x_{j,N})^2} \quad (j=1,2,\dots,N). \tag{3.32b}$$

The corresponding expression of u is represented by

$$u \sim \sum_{j=1}^N \frac{2\epsilon\beta_j t}{(x - \epsilon\alpha_j t^2)^2 + (\epsilon\beta_j t)^2}. \tag{3.33}$$

As an explicit example, we consider the case $N=2$. Since $H_4 = x^4 - 6x^2 + 3$ by (3.23), we obtain $x_{1,2}^2 = 3 + \sqrt{6}$, $x_{2,2}^2 = 3 - \sqrt{6}$ so that $\alpha_1 = \beta_1/2 = 1/[2(x_{1,2})^2] = (1 - \sqrt{2/3})/2$ and $\alpha_2 = \beta_2/2 = 1/[2(x_{2,2})^2] = (1 + \sqrt{2/3})/2$. These results coincide perfectly with (3.7).

In concluding this section it is worthwhile to remark that the asymptotic expression of u in the case of $\epsilon=0$ is given by²

$$u \sim \frac{2a_1}{(x_1 - a_1 t)^2 + a_1^2} + 2\pi \sum_{j=1}^{N-1} \delta(x - \xi_j), \tag{3.34}$$

where ξ_j ($j=1,2,\dots,N-1$) are zeros of the algebraic equation of degree $N-1$

$$\sum_{j=1}^{N-1} (-1)^j a_j x^{N-j} + (-1)^N a_N = 0. \tag{3.35}$$

Thus the presence of the damping changes drastically the properties of solutions.

IV. LINEARIZED EQUATION

In this section we solve the initial value problem of a linearized version of Eq. (1.1)

$$H\theta_t = -\theta - \epsilon\theta_x, \tag{4.1}$$

and compare the results with those obtained in Sec. III A. Equation (4.1) may be derived by linearizing Eq. (1.1) around the constant solutions $\theta = 2n\pi$ ($n=0, \pm 1, \dots$). The general solution of Eq. (4.1) can be represented by the Fourier integral in the form

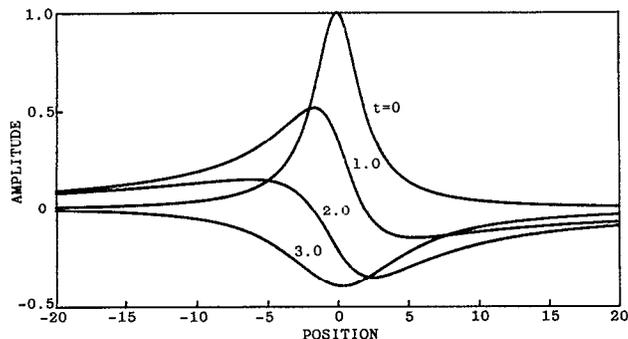


FIG. 11. The time evolution of the solution for a linearized damped sH equation.

$$\theta(x,t) = \int_{-\infty}^{\infty} \hat{\theta}(k) e^{-i(kx - \omega t)} dk, \tag{4.2a}$$

where the Fourier component $\hat{\theta}(k)$ is related to the initial value of θ as

$$\hat{\theta}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta(x,0) e^{ikx} dx, \tag{4.2b}$$

and $\omega = \omega(k)$ is a dispersion relation. The k dependence of ω is easily determined from (4.1) and (4.2) by using the formula $He^{-ikx} = -i \operatorname{sgn}(k) e^{-ikx}$ and it is written as

$$\omega = -\operatorname{sgn} k + i\epsilon |k|. \tag{4.3}$$

Substituting (4.2b) and (4.3) into (4.2a) and performing the integral with respect to k , one obtains the solution of Eq. (4.1) as follows:

$$\begin{aligned} \theta(x,t) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon t \cos t + (y-x) \sin t}{(y-x)^2 + (\epsilon t)^2} \theta(y,0) dy \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos t + y \sin t}{y^2 + 1} \theta(x + \epsilon t y, 0) dy. \end{aligned} \tag{4.4}$$

In order to visualize the profile of $u = \theta_x$, we specify the initial condition as

$$u(x,0) = \frac{4}{x^2 + 4}. \tag{4.5}$$

The functional form of (4.5) is the same as that of the initial profile of the one-soliton solution with $a_1 = 2.0$ and $b_1 = 0$ [see (3.1)]. In this case the y integral in (4.4) is easily performed by using the residue theorem to yield the result

$$u(x,t) = \frac{2(\epsilon t + 2) \cos t - 2x \sin t}{x^2 + (\epsilon t + 2)^2}. \tag{4.6}$$

Figure 11 shows the profiles of u with $\epsilon = 1.0$ at $t = 0, 1.0, 2.0, 3.0$. The figure may be compared with Fig. 1 which represents the nonlinear time evolution of the initial profile (4.5). In the linear approximation the peak position of the pulse goes back to its initial position, $x = 0$ in the present example, with a period 2π . The feature of wave phenomena is quite different from that of the nonlinear case where the corresponding position propagates to the right direction as time goes (see Fig. 1).

V. CONCLUDING REMARKS

In this paper we proposed a damped sH equation and showed that it can be solved exactly through a linearization procedure. The soliton solutions were then obtained and their properties were investigated in detail. It was found that the presence of the damping changes drastically the characteristics of solutions when compared with those of the sH equation. In particular the blowup of solutions was suppressed perfectly due to the effect of the damping. In this respect, however, it is quite interesting to remark that under certain situations an addition of the damping term makes the blowup sooner.¹¹

Although we have been concerned only with the non-periodic solutions throughout the paper, the periodic solutions will be constructed by a similar method. In this case it is appropriate to use Eq. (2.5) with nonzero λ instead of Eq. (2.7). This important problem will be dealt with elsewhere.

As already mentioned in Sec. I, the damped sH equation was introduced only from the mathematical point of view and hence at present it has no applications in physical phenomena. However, one may try to derive the equation on the basis of the fluid equation, the Navier-Stokes equation for instance under appropriate initial and boundary conditions.

ACKNOWLEDGMENT

I would like to thank Professor M. Nishioka for continual encouragement.

¹ Y. Matsuno, Phys. Lett. A **119**, 229 (1986).
² Y. Matsuno, J. Phys. A: Math. Gen. **20**, 3587 (1987).
³ Y. Matsuno, Phys. Lett. A **120**, 187 (1987).
⁴ Y. Matsuno, in *Proceedings of the Workshop on Finite Dimensional Integrable Nonlinear Dynamical Systems*, edited by P. G. Leach and W.-H. Steeb (World Scientific, Singapore, 1988), p. 205.
⁵ Y. Matsuno, J. Math. Phys. **33**, 412 (1992).
⁶ A. Degasperis and P. M. Santini, Phys. Lett. A **98**, 240 (1983).
⁷ A. Degasperis, P. M. Santini, and M. J. Ablowitz, J. Math. Phys. **26**, 2469 (1985).
⁸ P. M. Santini, M. J. Ablowitz, and A. S. Fokas, J. Math. Phys. **28**, 2310 (1987).
⁹ Y. Matsuno, *Bilinear Transformation Method* (Academic, New York, 1984), p. 178.
¹⁰ Yu. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. **61**, 763 (1989).
¹¹ Y. Matsuno, J. Math. Phys. **32**, 120 (1991).